Stochastic Process (II)

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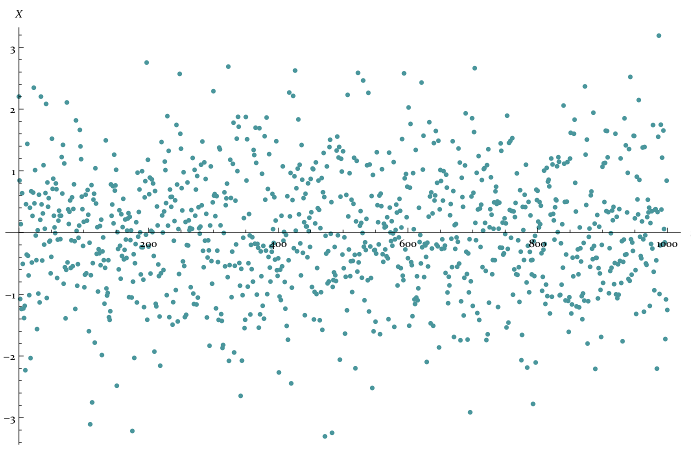
Autocorrelation

We have studied the PDF of random process *X(t)*.
A white-noise random variable $X(t_i)$ at t_i has a Gaussian PDF, and is independent on $X(t_{i-1})$, $X(t_{i-2})$, ...

In[6]:=
```
ListPlot[RandomFunction[WhiteNoiseProcess[], {0, 1000}], ImageSize → Large, AxesLabel → {t, X}]
```

Out[6]=



What if $X(t_i)$ depends on its history? -> "Autocorrelation" $\rho(t_i,\ t_{i-1})$

$0<\rho(t_i,\ t_{i-1})\leq 1$: $X(t_i)$ is more likely to be positive if $X(t_{i-1})$ is positive;
$-1\leq\rho(t_i,\ t_{i-1})<0$: $X(t_i)$ is more likely to be negative if $X(t_{i-1})$ is positive.

Stationary process:

$$\rho(t_i,\ t_{i-h}) = \rho(t_i - t_{i-h})$$

$$\approx \sum_{i=h+1}^{n} (X_i - \mu)(X_{i-h} - \mu) / \sum_{i=1}^{n} (X_i - \mu)^2$$

e.g., for a continuous white noise, $\rho(t_i - t_{i-h}) \approx \rho(h) = \delta(h)$.

```
CorrelationFunction[WhiteNoiseProcess[], i - h, i]
DiscreteDelta[h]
```

Spectrum density

```
FourierTransform[DiracDelta[τ], τ, ω]
```

$$\frac{1}{\sqrt{2\pi}}$$

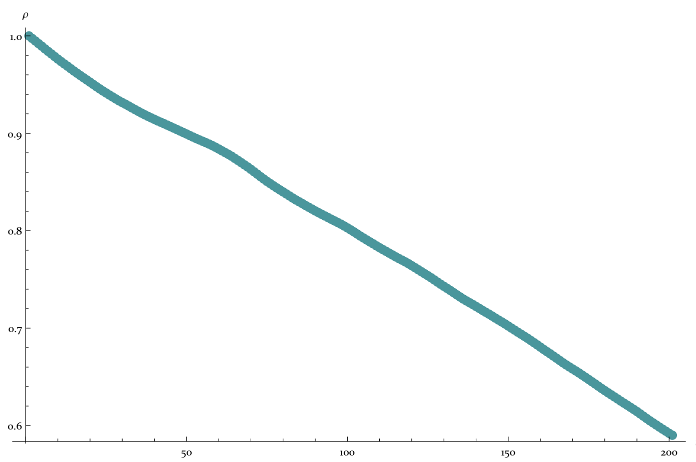
Wiener process is not stationary, but has stationary increments; it is called a Markov process. However, one can still try to “calculate” its correlation function using the formula for stationary process.

```
CorrelationFunction[WienerProcess[], t1, t2]
```

$$\frac{\text{Min}[t1, t2]}{\sqrt{t1\, t2}}$$

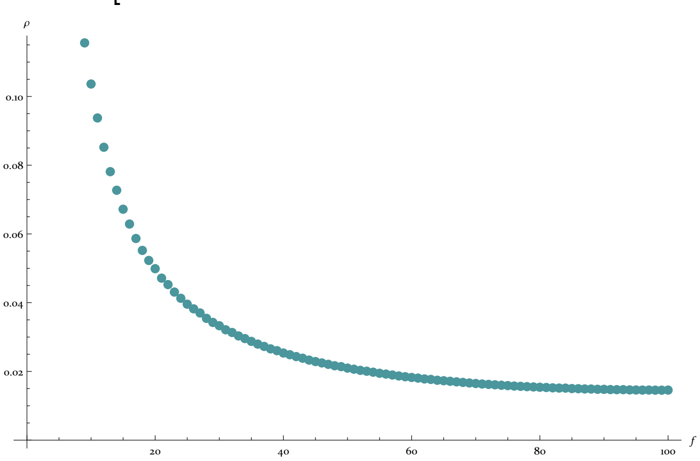
```
In[7]:= ClearAll["Global`*"]
data = RandomFunction[WienerProcess[], {0, 2000, 1}]["ValueList"][[1]];
corr = CorrelationFunction[data, {200}];
ListPlot[corr, ImageSize → Large, AxesLabel → {t, ρ}]
```

Out[10]=




```
In[12]:= ListPlot[Abs@Fourier[corr][[1 ;; 100]], ImageSize → Large, AxesLabel → {f, ρ}]
```

Out[12]=



Wiener process is also called $1/f^2$ noise, aka, Brown noise.

Fractional white noise & fractional Brownian motion

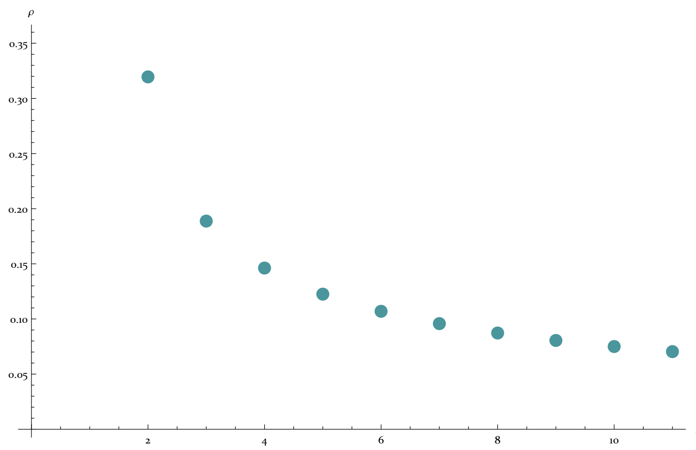
Fractional white noise also has Gaussian PDF, but non-zero autocorrelation.

Hurst exponent *H*: *H*=1/2 corresponds to Gaussian noise.

```
In[13]:= CorrelationFunction[FractionalGaussianNoiseProcess[H], i - h, i]
ListPlot[Table[% /. {H → 0.7}, {h, 0, 10}], ImageSize → Large, AxesLabel → {t, ρ}]
```

Out[13]= $\frac{1}{2}\left(\text{Abs}[-1+h]^{2H} - 2\,\text{Abs}[h]^{2H} + \text{Abs}[1+h]^{2H}\right)$

Out[14]=



Integrating fractional white noise yields fractional Brownian motion.

```
FractionalBrownianMotionProcess[μ, σ, H][t]
NormalDistribution[t μ, t^H σ]
```

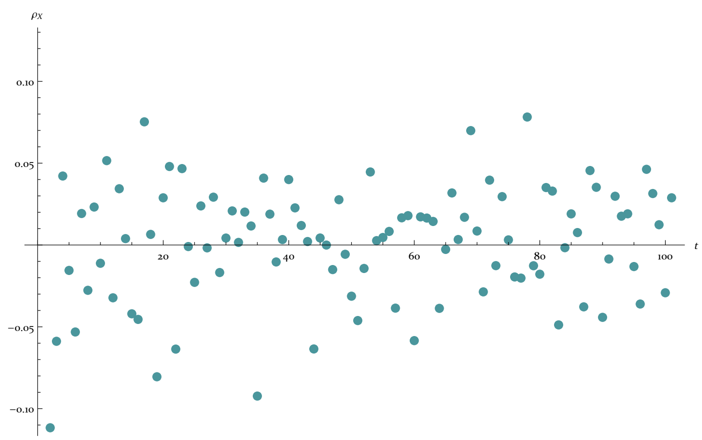
Levy flight vs Fractional Brownian motion?

Levy flight shows the scaling behavior of PDF correctly; but it has infinite variance σ. On the other hand, the fractional Brownian motion has finite variance and non-trivial scaling behavior, but its PDF is only Gaussian.

Scaling of autocorrelation in stock market

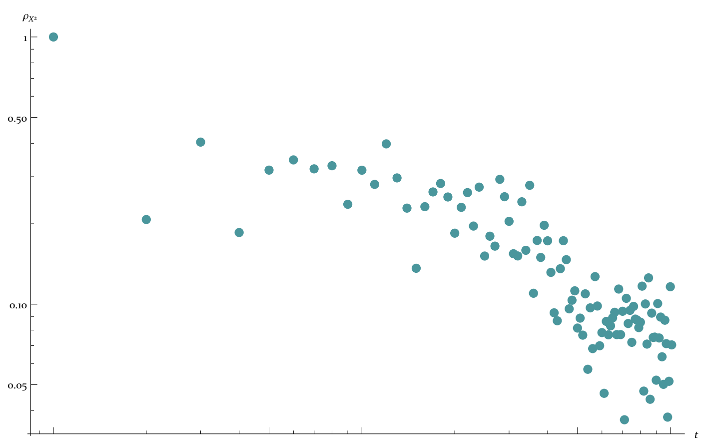
In[23]:=
```
ClearAll["Global`*"];
sp500 = FinancialData["SP500", {{2006}, {2015}}, "Value"];
CorrelationFunction[Differences[Log[sp500]], {100}];
ListPlot[%, ImageSize → Large, AxesLabel → {t, "ρX"}]
```

Out[26]=



In[27]:=
```
corrVolatility = CorrelationFunction[Differences[Log[sp500]]^2, {100}];
ListLogLogPlot[corrVolatility, ImageSize → Large, AxesLabel → {t, "ρX²"}]
(*FindFit[corrVolatility~Drop~1,A t^δ,{A,δ},t]*)
```

Out[28]=



AR, ARCH, and GARCH processes

(x[t] is log return)

AR(1): $x[t] = a_0 + a_1\, x[t-1] + \epsilon[t]$, ϵ[t] = σ z[t], z[t]~Gaussian[0,1];
->autocorrelation of returns

ARCH(1): $x[t] = (a_0 + a_1\, x[t-1] + \ldots) + \epsilon[t]$, ϵ[t]=σ[t]z[t],
$\sigma^2[t] = \alpha_0 + \alpha_1\, \epsilon^2[t-1]$ z[t]~Gaussian[0,1];
->autocorrelation of volatility

GARCH(1,1): $x[t] = (a_0 + a_1\, x[t-1] + \ldots) + \epsilon[t]$, ϵ[t]=σ[t]z[t],
$\sigma^2[t] = \alpha_0 + \alpha_1\, \epsilon^2[t-1] + \beta_1\, \sigma^2[t-1]$ z[t]~Gaussian[0,1];
->autocorrelation of volatility

```
ClearAll["Global`*"];
sp500returns = Differences@Log@FinancialData["SP500", {{2006}, {2015}}, "Value"];
EstimatedProcess[sp500returns, ARProcess[1], ProcessEstimator -> "MethodOfMoments"]
(*RandomFunction[%,{1,1000}]["ValueList"][[1]];
ListPlot[{Exp@Accumulate[%]}]*)
ARProcess[0.000237701, {-0.111661}, 0.000177524]
```

GARCH process

The asymptotic variance and kurtosis (4-th moments, ≈3 for Gaussian PDF) of GARCH(1,1) are $\frac{\alpha_0}{1-\alpha_1-\beta_1}$ and $3 + (6\,\alpha_1{}^2)/(1 - 3\,\alpha_1{}^2 - 2\,\alpha_1\,\beta_1 - \beta_1{}^2)$.

```
Variance[GARCHProcess[α0, {α1}, {β1}][t]]
```

$$\frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

```
Kurtosis[GARCHProcess[α0, {α1}, {β1}][t]]
```

$$\begin{cases} \text{Indeterminate} & \alpha_1 \le 0 \;||\; \beta_1 \le 0 \;||\; 3\,\alpha_1^2 + 2\,\alpha_1\,\beta_1 + \beta_1^2 \ge 1 \\ (3\,(-1 + \alpha_1 + \beta_1)\,(1 + \alpha_1 + \beta_1)) \Big/ \left(-1 + 3\,\alpha_1^2 + 2\,\alpha_1\,\beta_1 + \beta_1^2\right) & \text{True} \end{cases}$$

However, the autocorrelation for $x^2[t]$ is $\rho(\tau) = e^{-|\mathrm{Ln}(\alpha_1+\beta_1)|\tau}$, not power-law but short-range. Also, the GARCH model is only **discrete.**

Conclusion

- **1) . There is also scaling of autocorrelation (of volatility) in the stock market.**
- **2) . The Fractional Brownian motion can yield non-zero autocorrelation of returns, as well as non-trivial scaling exponent for PDFs.**
- **3) . The GARCH model does the best job. It's only a discrete model, though.**

Can we find a continuous limit of the GARCH model?

(Surprisingly, only some naive results have been found so far, in which at least two indepen dent Wiener processes are required to yield a correct limit.)