Options and The Black-Scholes-Merton Model

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Outline

- Motivation
- Options
- Review of the Stock Price
- The BSM model
  - Assumptions
  - Derivation
  - Solution for European Options
- The Monte Carlo method for European Options

Motivation

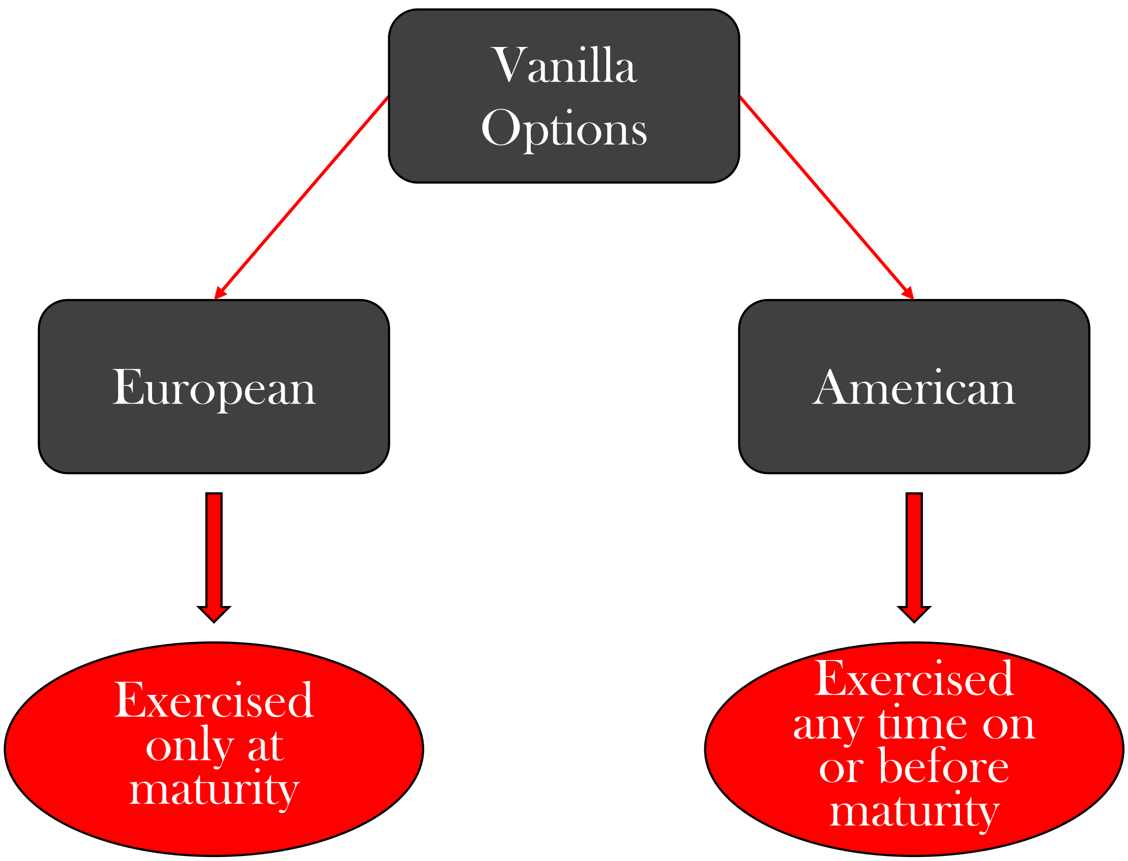
- Introduction to Options
- Introduction to the BSM model.
  - Is it enough?
  - Are there any improvements?
- How do we valuate an option using Monte Carlo?

Options

Basic securities include stocks, bonds and mutual funds.

Options are more sophisticated securities.
In general,
"*An option is a contract which gives the buyer the right, but NOT the obligation to buy or sell an underlying asset or instrument at a specified strike price, on or before a specified date.*"

Option Styles



A few other options are

- Bermudan
- Asian
- Game
- Barrier
- Binary
- Basket
- Rainbow
- Other Exotic options

Basic Option Terminology

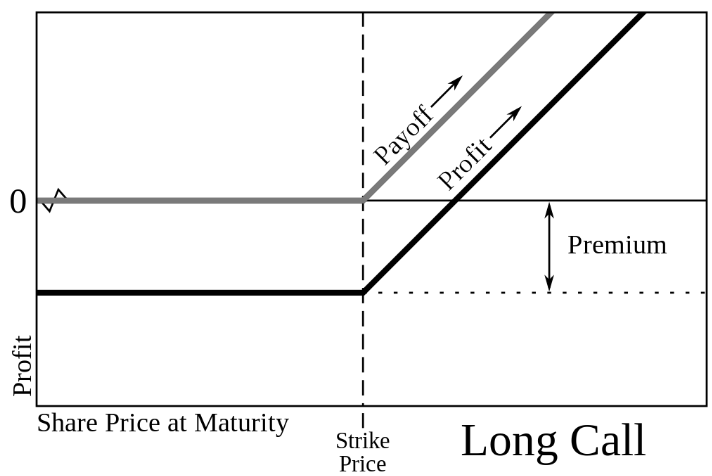
- Maturity
  Is the date at which the option can be exercised.

- Strike Price
  Is the price at which a specific derivative can be exercised.

- Premium
  Is the cost of an option.

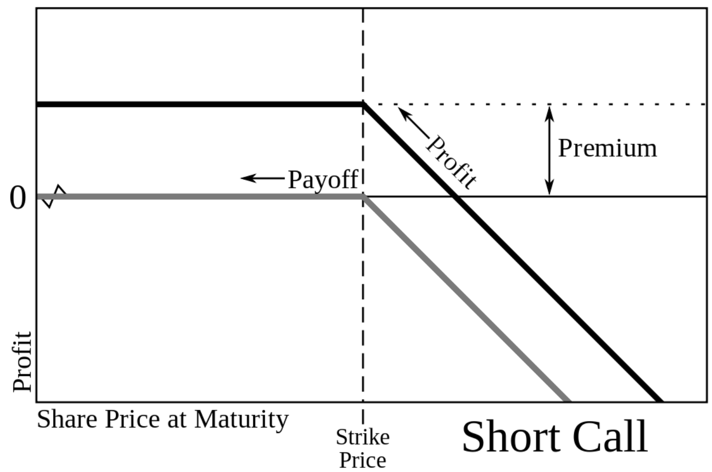
- Payoff
  The received amount of money by the option.

Long Call Option



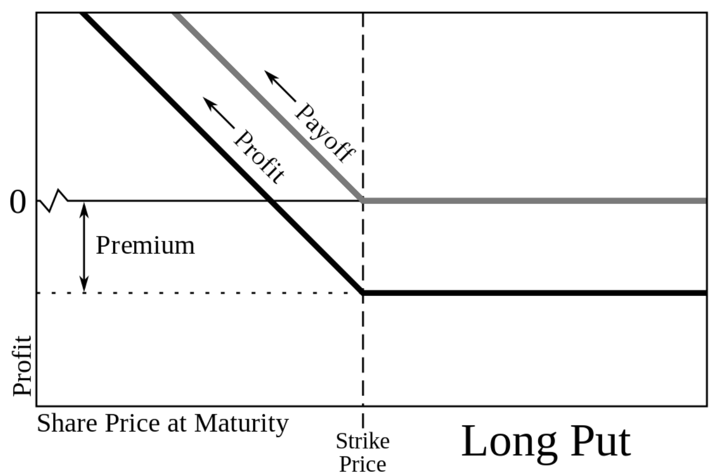
In this case we pay a premium and we buy the right to buy the underlying asset at the strike price.

Short Call Option



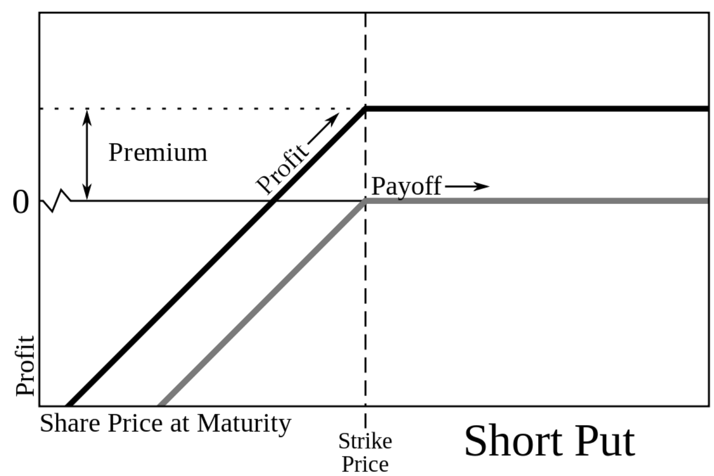
In this case we receive a premium and we give the right to someone else to buy from us the underlying asset at the strike price.

Long Put Option



In this case we pay a premium and we buy the right to sell the underlying asset at the strike price.

Short Put Option



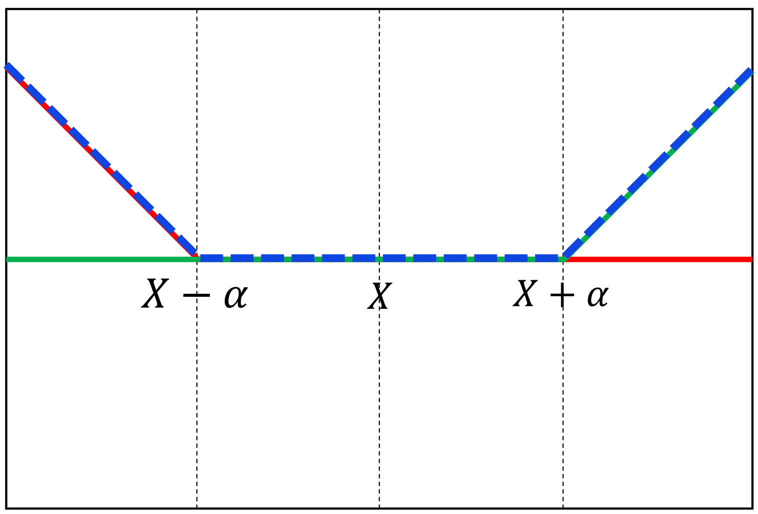
In this case we receive a premium and we give the right to someone else to sell us the underlying asset at the strike price.

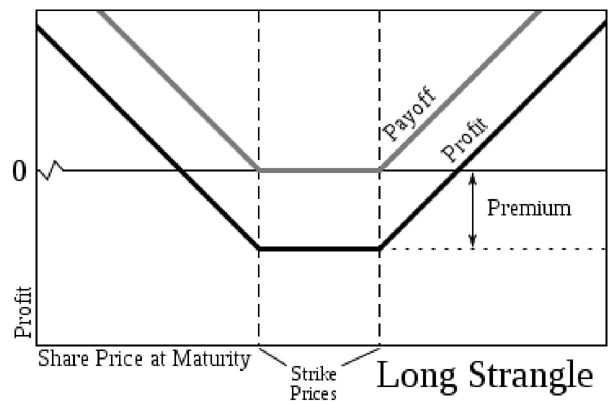
These four can be used as building blocks for other much more sophisticated strategies.

Some examples of such strategies are

- Butterfly
- Strangle
- Straddle
- Collar
- Iron Condor
- Other fancy names…

Strangle (bet on high volatility)

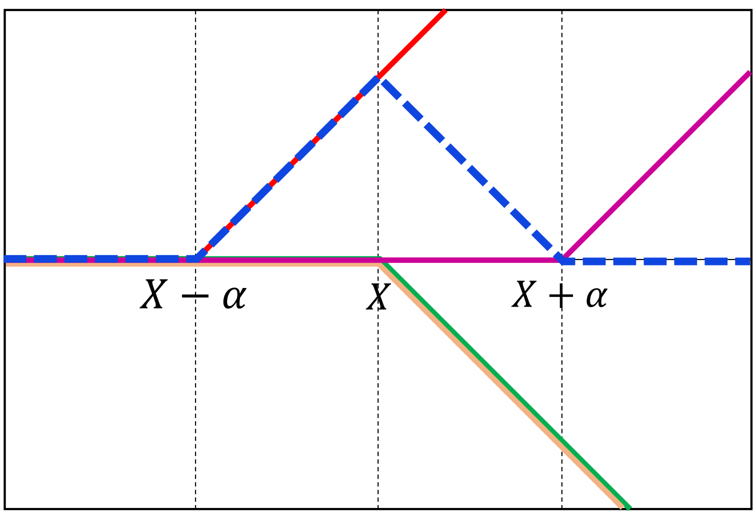


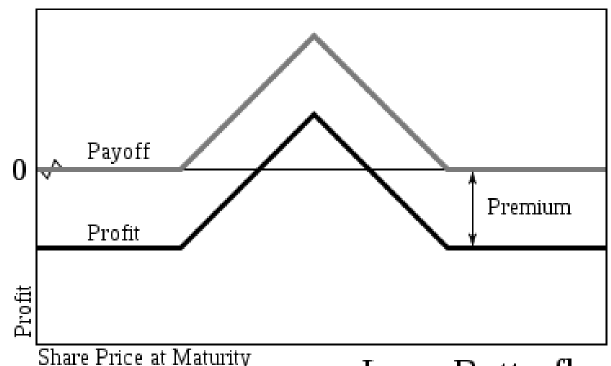


1 x Long put with strike price $X - \alpha$

1 x Long call with strike price $X + \alpha$

Butterfly (bet on low volatility)





Long Butterfly

1 x Long call with strike price $X - \alpha$
2 x Short calls with strike price X
1 x Long call with strike price $X + \alpha$

Trading options is VERY risky business...
Assume a stock with initial price $S_0 = 100$ and a call option with strike price of $K = 105$ and premium $P = 5$.

Stock Price	Stock Return	Option Payoff	Option Return
95	-5%	0	-100%
100	0%	0	-100%
105	5%	0	-100%
107	7%	2	-60%
110	10%	5	0%
115	15%	10	100%
120	20%	15	200%

Why do people use options?

- Speculation
- Hedging

Where can you trade options?

- CBOE (Chicago Board Options Exchange)
- Over-the-counter

Who trade options?

- Hedge Funds
- Investment Banks
- Big organizations
- Some individuals

Review on the Stock Price

We assume the process

$$dS = \mu S dt + \sigma S dz$$

The *percentage changes* in the stock price in a small period of time are normally distributed.

$$\frac{\Delta S}{S} \sim \phi\left(\mu \Delta t, \sigma\sqrt{\Delta t}\right)$$

Which leads to

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

Since $\ln S_T$ is normally distributed, S_T is lognormally distributed.

What is the lognormal distribution?
Assume X follows a normal distribution with PDF

$$N(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

By definition, the lognormal distribution $LN[\mu,\sigma^2]$ is the distribution of e^X.

Some propositions:

1. If X is $N[\mu, \sigma^2]$ then $E(X) = \mu$ and $Var(X) = \sigma^2$.

2. If Y is $LN[\mu, \sigma^2]$ then $E(Y) = e^{\mu + \frac{\sigma^2}{2}}$, $Var(Y) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.

3. If X is $N[\mu, \sigma^2]$ then $aX + b$ is $N[a\mu + b, a^2\sigma^2]$.

4. If X_1 and X_2 are independent and X_1 is $N[\mu_1, \sigma_1^2]$ and X_2 is $N[\mu_2, \sigma_2^2]$, then X is $N[\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2]$ where $X = X_1 + X_2$.

5. The PDF of $LN[\mu, \sigma^2]$ is $\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$.

Proof of 1

$$E(X) = \int_{-\infty}^{+\infty} xPDF \, \mathrm{d}x \quad = \mu$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 PDF \, \mathrm{d}x \quad = \mu^2 + \sigma^2$$

$$Var(X) = E(X^2) - E(X)^2 = \sigma^2$$

Proof of 2

$$E(Y) = E(e^X) = \int e^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x$$

$y = x - \mu$

$$= \int e^{y+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}\,\mathrm{d}y$$

$$= e^{\mu} \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2-2\sigma^2 y+\sigma^4-\sigma^4}{2\sigma^2}}\,\mathrm{d}y$$

$$= e^{\mu+\frac{\sigma^2}{2}} \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\sigma^2)^2}{2\sigma^2}}\,\mathrm{d}y \quad 1$$

$$= \boxed{e^{\mu+\frac{\sigma^2}{2}}}$$

For $E(Y^2)$ we need $E(e^{2X})$ and in this case we get

$$E(Y^2) = e^{2\mu+2\sigma^2}$$

So the variance is

$$Var(Y) = E(Y^2) - E(Y)^2$$

$$= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$$

$$= \boxed{e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)}$$

It follows from (4) that

- If X_i are independent $N[\mu, \sigma^2]$, then $\sum_{i=1}^{n} X_i$ is $N[n\mu, n\sigma^2]$
- If Y_i are independent $LN[\mu, \sigma^2]$, then $\prod_{i=1}^{n} Y_i$ is $LN[n\mu, n\sigma^2]$.

In order to find the PDF of the LN we assume that

$$Y \sim \mathrm{LN}[\mu, \sigma^2]$$

Then, by definition, $Y = e^X$ where $X \sim N[\mu, \sigma^2]$.

Hence,

$$\begin{aligned} Prob(Y < k) &= Prob(e^X < k) \\ &= Prob(X < \ln k) \\ &= \int_{-\infty}^{\ln k} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, \mathrm{d}x \end{aligned}$$

$$y = e^x, \qquad x = \ln y, \qquad \mathrm{d}x = \frac{1}{y}\mathrm{d}y$$

$$= \boxed{\int_{-\infty}^{k} \frac{1}{y}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}\, \mathrm{d}y}$$

For the stock price we know

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

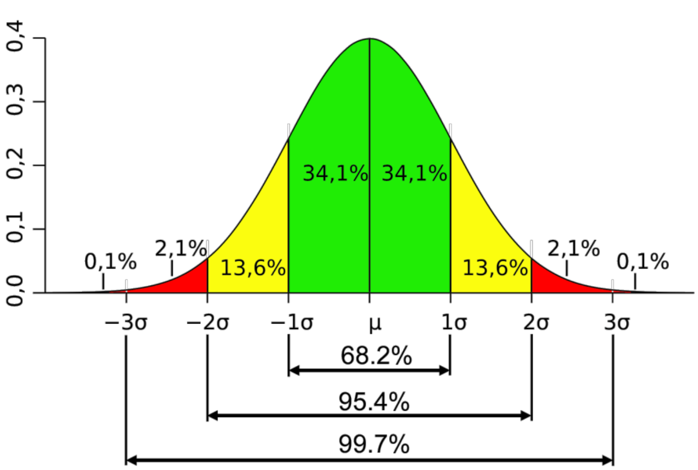
So, $\tilde{\mu} = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T$ and $\tilde{\sigma} = \sigma\sqrt{T}$.

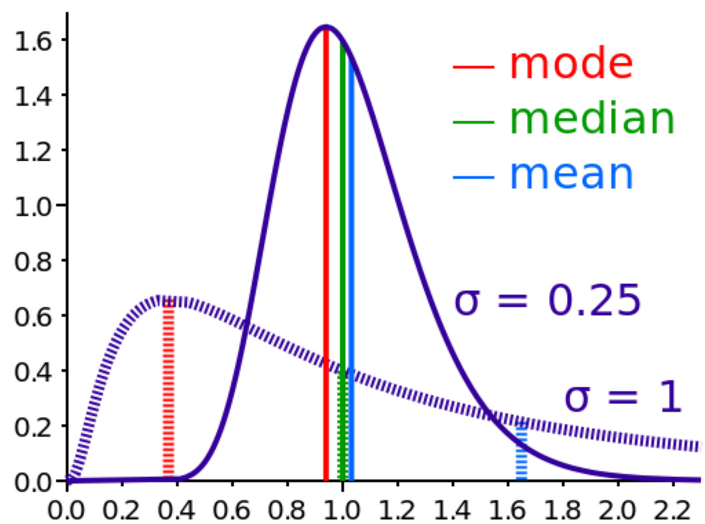
For the LN distribution we proved that $E(Y) = e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}}$ and $Var(Y) = e^{2\tilde{\mu}+\tilde{\sigma}^2}\left(e^{\tilde{\sigma}^2} - 1\right)$.

So,

$\boxed{E(S_T) = S_0 e^{\mu T}}$ and $\boxed{Var(S_T) = S_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)}$

Normal vs Lognormal





The expected return μ

The value of μ should depend on the risk of the investment as well as the interest rates in the economy.
It turns out that the value of a derivative which depends on a stock is, in general, independent of μ.

What is μ?
It's the expected change in the stock price, $\Delta S/S$, during a period of time, ΔT.

Now let's define the continuously compounded rate (CCR) of return per annum, r, realized on a stock between times 0 and T. Then,

$$\boxed{S_T = S_0 e^{rT}}$$

Remember also that $E(S_T) = S_0 e^{\mu T}$.

The 1 million dollar question: What is r?

It turns out that μ is NOT the expected CCR on the stock !!!

By definition, $r = \frac{1}{T}\ln\left(\frac{\mathrm{S_T}}{\mathrm{S_0}}\right) = \frac{\ln S_T}{T} - \frac{\ln S_0}{T}$

But since we know $\ln S_T \sim \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$
We get

$$r \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}}\right)$$

So $E(r) = \mu - \frac{\sigma^2}{2}$!

Why are they different?

Imagine you are the manager of a very big Hedge Fund and you want to report on how well your strategy performs annually. Your returns over the last 5 years are

15%, 20%, 10%, -20%, 25%

What number should you choose?

First notice that the number we are looking for after 5 years is $S_5 = (1.15 \times 1.2 \times 1.1 \times 0.8 \times 1.25)S_0 = 1.52S_0$

If we use the average $(15 + 20 + 10 - 20 + 25)/5 = 10\%$ and compound for 5 years $1.1^5 S_0 = 1.61S_0$

But if we use the geometric average $\sqrt[5]{1.15 * 1.2 * 1.1 * 0.8 * 1.25} = 1.087$ and $1.087^5 = 1.52$.

The Volatility σ

Usually we use historical data of the stock price. Given that the stock price is S_i, then we define the log-returns as

$$x_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

and the volatility as the standard deviation of x_i.

$$v = \sqrt{\frac{1}{n-1}\sum (x_i - \bar{x})^2}$$

$$= \sqrt{\frac{1}{n-1}\sum x_i^2 - \frac{1}{n(n-1)}\left(\sum x_i\right)^2}$$

The Black-Scholes-Merton Model

1973: Fisher Black and Myron Scholes publish their paper "The Pricing of Options and Corporate Liabilities".

1973: Robert Merton publishes his paper "Theory of Rational Option Pricing".

1997: Merton and Scholes are awarded the Nobel prize for economics. Black could not receive it since he died in 1995.

Their papers were two of the most seminal papers in the history of Finance.

Assumptions of the BSM model

1. The stock price follows the Itô process described before

2. No commissions

3. No dividends are paid

4. Buy and sell of any amount of the stock is allowed

5. The risk-free rate and volatility of the underlying are known and constant

6. No arbitrage

7. The trading is continuous

Itô's Lemma

If a variable x follows the Itô process

$$x = \alpha(x,t)dt + \beta(x,t)dz$$

then a function $f = f(x,t)$ follows the process

$$df = \left(\frac{\partial f}{\partial x}\alpha + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\beta^2\right)dt + \frac{\partial f}{\partial x}\beta dz$$

In the case of $dS = \mu S dt + \sigma S dz$, using the Itô's lemma for a function $f(S,t)$ we can write

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dz$$

or

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z$$

f can be the price of the derivative.

Consider the portfolio

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

This portfolio is short one derivative and long $\partial f / \partial S$ shares. This is a riskless portfolio because the gains or losses from the derivative always balance the gains or losses from the stock position.

A change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

Substituting Δf and ΔS we get

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

But remember that since this is a riskless portfolio, the return of this portfolio should be given by

$$\Delta\Pi = r\Pi\Delta t$$

Substituting this to the above equation we get the famous BSM equation

$$\boxed{\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf}$$

Risk Neutral Valuation

The BSM equation has NO variables that depend on the risk preferences of the investors.

That means that for r we can use ANY value we want, as long as we discount the expected payoff at this specific value.

BSM for European Options

The boundary conditions are

$f_{call,T} = \max(S - K, 0)$ and $f_{put,T} = \max(K - S, 0)$

For a non-dividend paying stock, the price at time 0 is

$$\boxed{C = S_0 N(d_1) - Ke^{-rT} N(d_2)}$$

and

$$\boxed{P = -S_0 N(-d_1) + Ke^{-rT} N(-d_2)}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

$N(x)$ is the cumulative probability distribution function for $\phi(0,1)$

Connection to Physics

With proper change of variables, the BSM can be reduced to an equation of the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Do you recognize this equation?

It’s the heat equation in one dimension

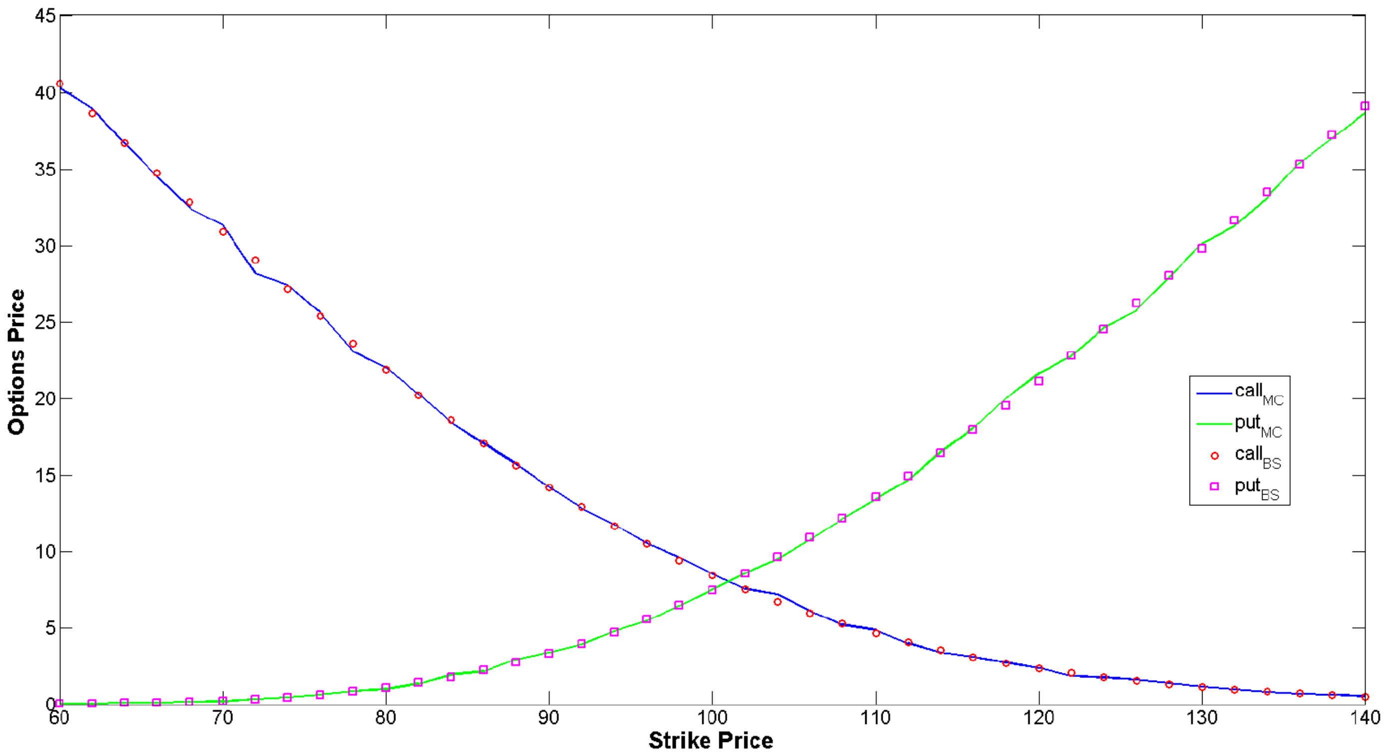
Monte Carlo for European Options

The MC method is divided in three main steps

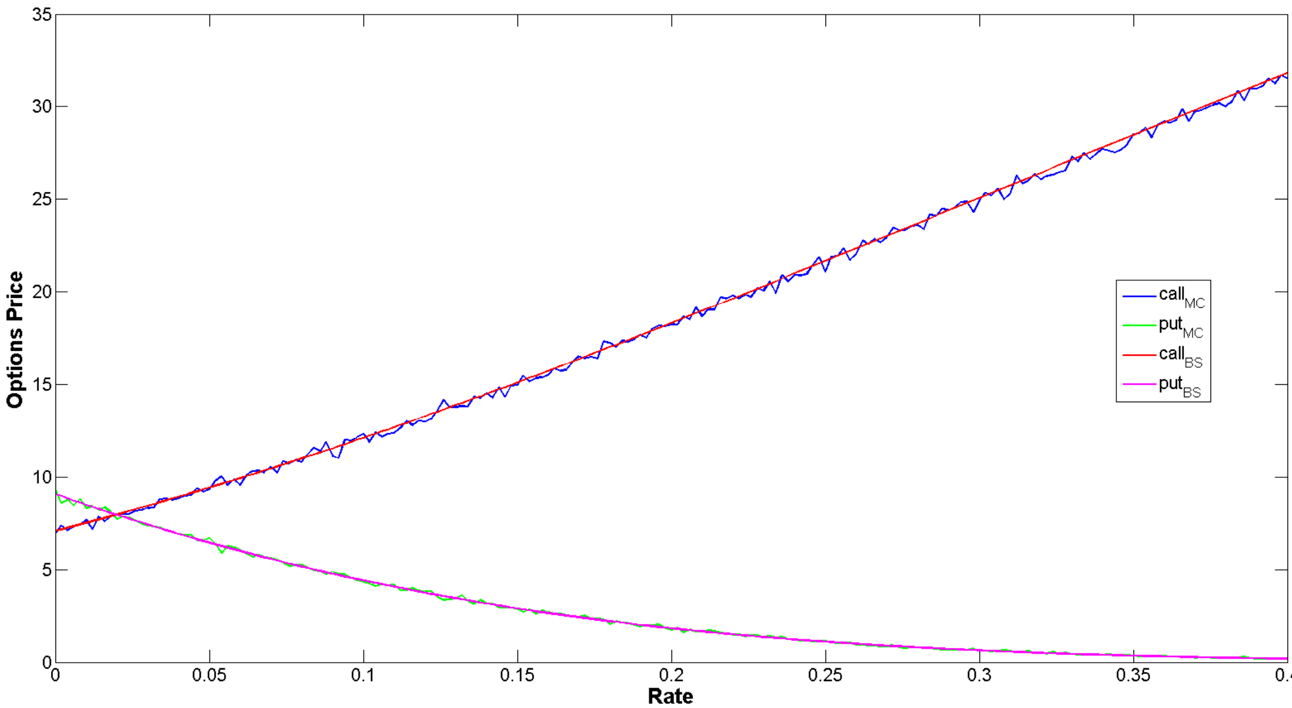
1. Calculate future price using random walk
2. Calculate the payoff for this price
3. Discount the payoff back to today

We repeat the above procedure for a reasonable number of steps and we average the results.

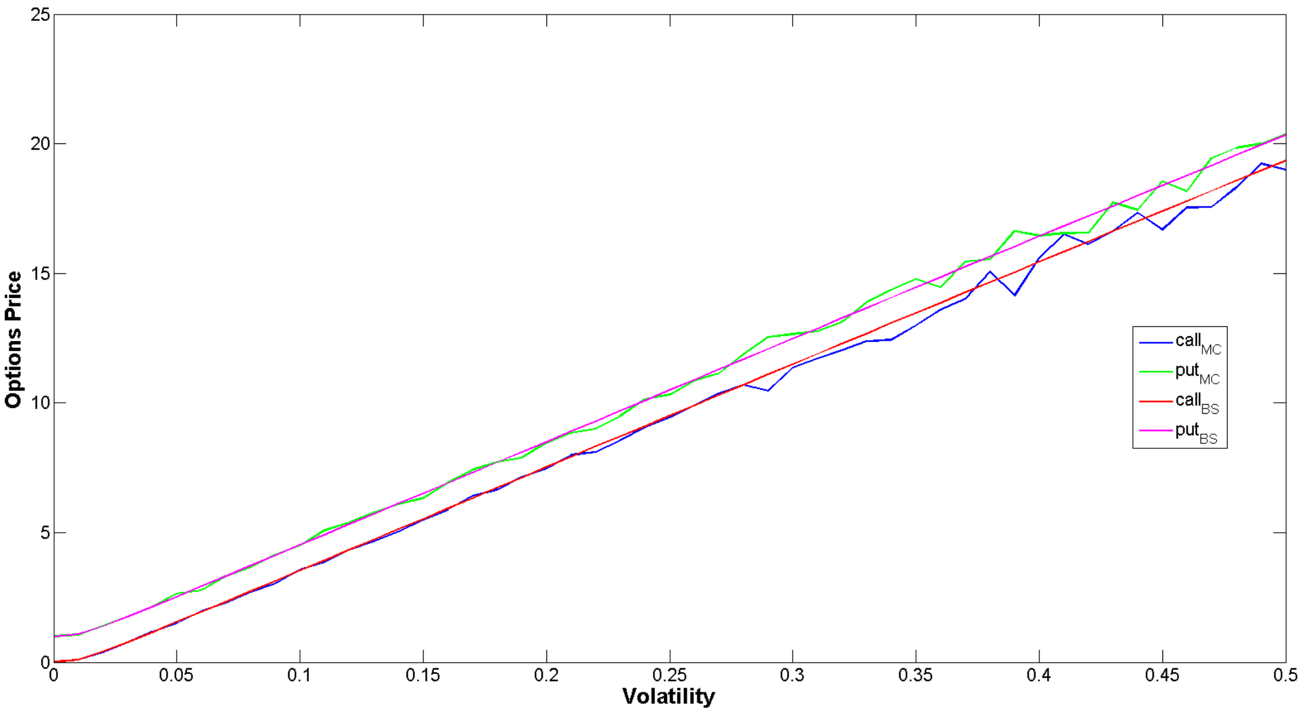
Option Price vs Strike Price



Option Price vs Interest Rate



Option Price vs Volatility



Numerical Methods

The BSM cannot always be solved analytically. Numerical methods to solve the BSM include

- Numerical methods to solve PDEs
- Monte Carlo
- Binomial or trinomial trees

Bibliography

[1] J. C. Hull, *Options, Futures and Other Derivatives*, 9th ed., Prentice Hall, 2014