# Random Walks in Physics and Finance

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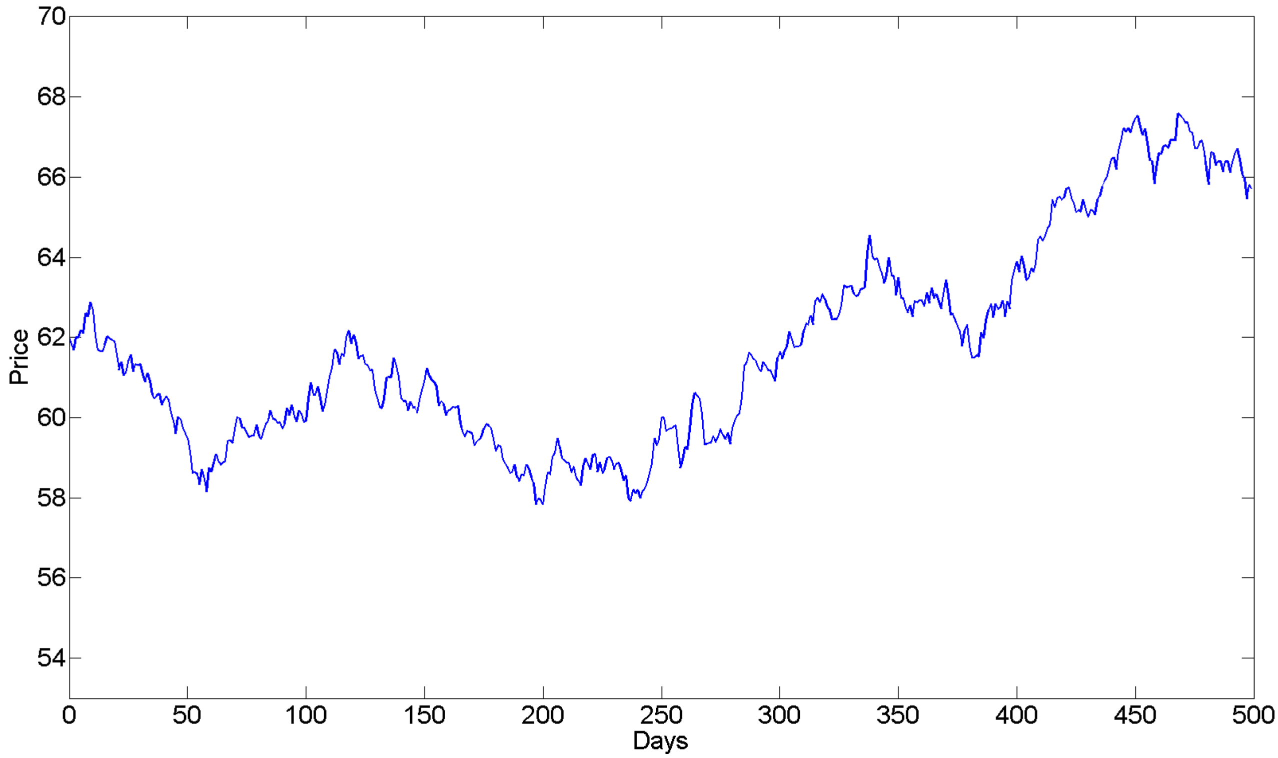
# What is this?



Price
Days
70
68
66
64
62
60
58
56
54
0
50
100
150
200
250
300
350
400
450
500

# Outline

- Motivation
- Bachelier's theory
- Random walks in Physics
- Wiener Process and Itô's Lemma
  - Wiener Process
  - Generalized Wiener Process
  - Ito Process
  - Process for a Stock Price
  - Itô's Lemma
  - The Lognormal Property

# Motivation

- How was the random walk introduced in Physics?
- Have random walks been observed in Physics?
- Can we describe the dynamics of the prices of financial assets?
- Is a random walk a realistic model for the stock market?
- Can one make predictions for price movements of securities and derivatives?

# “Théory de la Spéculation”

## Louis Bachelier, 1900

The goal was to predict the future price of a financial instrument given its current spot price

↓

Postulate

At any given moment, the market is neither bullish nor bearish

## Market Hypotheses

- Successive price movements are statistically independent
- In a perfect and efficient market, the current price is a result of all the available information from the past to the present and all stocks trade at their fair value
- In a complete market, there are both buyers and sellers at any quoted price

Derivation of the Random Walk

Consider two mutually exclusive events $A$ and $B$ happening with probability $p$ and $q = 1 - p$ respectively. These events represent the price movement by $\pm x_0$, in one step time.

The probability to observe $a$ realizations of $A$ in $m$ events is given by the binomial distribution

$$P = \frac{m!}{a!\,(m-a)!} p^a q^{m-a}$$

For given $m$ and $p$, which $a$ maximizes $P$?

Answer: $a = mp \implies m - a = mq$

So after $m$ trading days, the final price will most likely be

$$\begin{aligned} x_{max} &= a x_0 - (m-a) x_0 \\ &= mp x_0 - mq x_0 \\ &= m x_0 (p - q) \end{aligned}$$

If $p \neq q$ then that would suggest a drift in the market.
In the limit of $m, a \to \infty$ with $h = a - mp$, the distribution function becomes

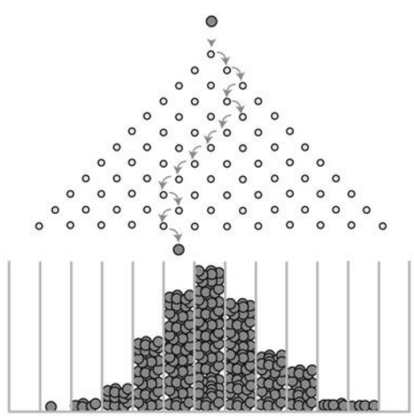
$$p(h) = \frac{1}{\sqrt{2\pi mpq}} e^{-\frac{h^2}{2mpq}}$$

Complete derivation https://en.wikipedia.org/wiki/De_Moivre-Laplace_theorem

Assuming that $p = q = 1/2$, $m = t/\Delta t$ and changing $h \to x$ we get

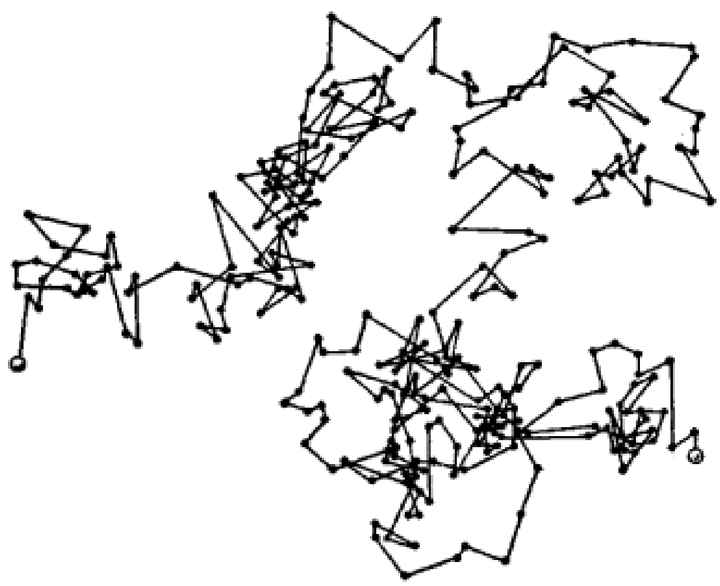
$$p(x) = \sqrt{\frac{2}{m\pi}} \exp\left(-\frac{2\pi x^2}{m}\right)$$

Notice that this is no longer discrete time! ($m \gg 0$ )

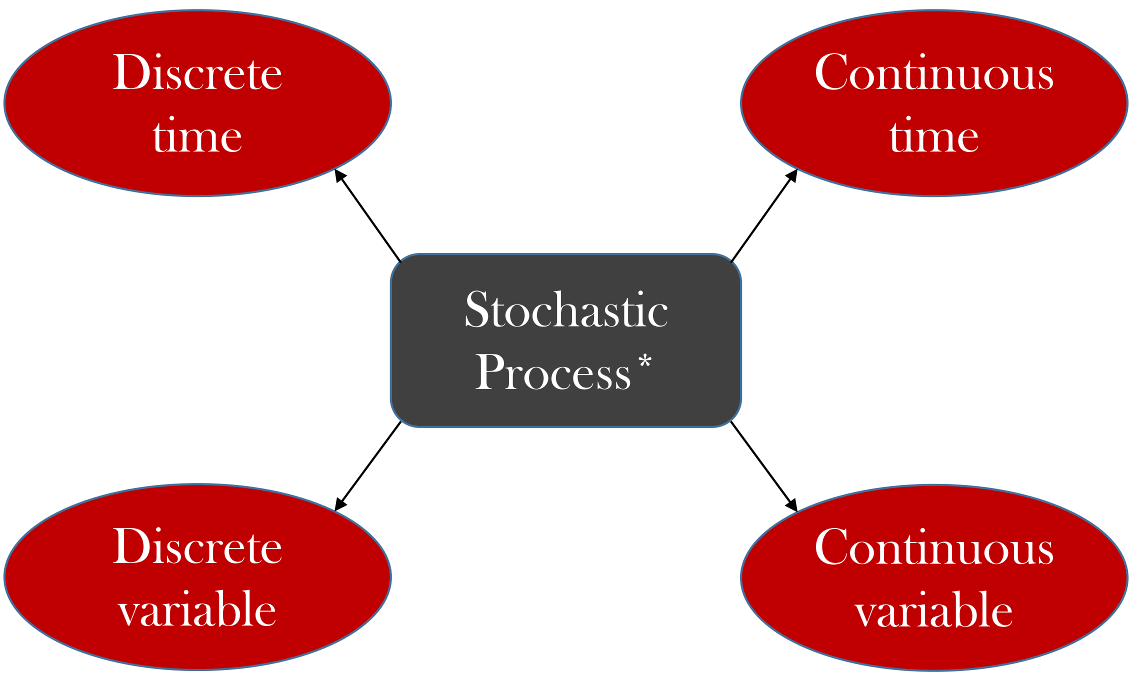


# Random walks in Physics

The French physicist Jean Perrin in 1909 observed the motion of colloidal particles of radius $0.53\ \mu m$ suspended in a liquid.



# Wiener Process and Itô's Lemma




*A variable whose value changes in an uncertain way over time is said to follow a stochastic process

The Markov Property

A stochastic process where *only the present value* of a variable is relevant for predicting the future is called Markov process.

These predictions are uncertain and must be expressed in terms of probability distributions.

The Markov property implies that the probability distribution of the price at any particular future time is independent of the path followed so far.

What if it was??? What if people could predict the future by looking at the past?

Continuous Time Stochastic Process

Suppose $X$ follows a Markov stochastic process. $X$ can be $10$ if you want. It is given that the change in $X$ in one year follows $\phi(0,1)$ where $\phi(\mu, \sigma)$ is the normal distribution with mean $\mu$ and variance $\sigma^2$.

What is the probability distribution of the change in $X$ in two years?

$$\phi\left(0, \sqrt{2}\,\right)$$

When adding two independent normal distributions

$$\mu_{tot} = \mu_1 + \mu_2$$

$$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2$$

In general, the change during any time period of length T follows

$$\phi\left(0, \sqrt{T}\right)$$

So, if the variance rate is 1 per year,
what about the change after six months?

$$\phi\left(0, \sqrt{0.5}\right)$$

What about after three months?

$$\phi\left(0, \sqrt{0.25}\right)$$

## Wiener Process

A Wiener process is the special case of a Markov process in which the mean change is $\mu = 0$ and $\sigma = 1$. It's also called *Brownian Motion*.

If $z$ follows a Wiener process

- The change $\Delta z$ during a small period of time $\Delta t$ is
$$\Delta z = \epsilon\sqrt{\Delta t}$$
Where $\epsilon \sim \phi(0,1)$. It follows from this that
$$\Delta z \sim \phi\left(0, \sqrt{\Delta t}\right)$$

- The values of $\Delta z$ for different time intervals $\Delta t$ are independent.

Example of a Wiener Process

If $z$ follows a Wiener process and $z_0 = 25$, then after one year

$$z_1 = 25 \pm 1$$

After five years,

$$z_2 = 25 \pm \sqrt{5}$$

Generalized Wiener Process

*Drift rate* is the mean change per unit time for a stochastic process, while *variance rate* is the variance per unit time.

The basic Wiener process, $dz$, has a drift rate of zero and a variance rate of one. What does that mean?

A generalized Wiener process for a variable $x$ is defined in terms of $dz$ as

$$dx = \alpha dt + \beta dz$$

where $\alpha$ and $\beta$ are constants.

Disregarding the second term we get

$$dx = \alpha dt$$

and by integrating we get

$$x = x_0 + at$$

which is a straight line indicating that in a period of time of length $T$, the variable $x$ changes by an amount $\alpha T$.

The $\beta dz$ term can be regarded as adding noise to the path followed by $x$.

Eventually, we can write

$$\Delta x = \alpha \Delta t + \beta \epsilon \sqrt{\Delta t}$$

which means that

$$\Delta x \sim \phi\left(\alpha \Delta t, \beta \sqrt{\Delta t}\right)$$

$$\alpha \rightarrow \text{drift rate, } \beta^2 \rightarrow \text{variance rate}$$

Example of a Generalized Wiener Process

A company's cash position in thousands of dollars follows a generalized Wiener process with a drift rate of $20$ per year and a variance rate of $900$ per year. The initial cash position is $50$. What happens after six months?

We identify $\alpha = 20$ and $\beta^2 = 900$. After six months the mean will be $50 + 10 = 60$ and the standard deviation will be $\sqrt{900}\sqrt{0.5} = 21.21$.
So,

$$x_{1/2} = 60 \pm 21.21$$

## Itô Process

Taking the idea of the generalized Wiener process even further, we allow $\alpha$ and $\beta$ to be functions of $x$ and $t$. Hence,

$$dx = \alpha(x, t)dt + \beta(x, t)dz$$

Assuming that $\alpha$ and $\beta$ remain constant in the small time interval between $t$ and $t + \Delta t$, the variable $x$ changes from $x$ to $x + \Delta x$ with

$$\Delta x = \alpha(x, t)\Delta t + \beta(x, t)\epsilon\sqrt{\Delta t}$$

# Quiz Time

The Process for a Stock Price

What could it be?

- Wiener
- Generalized Wiener
- Itô

It's an Itô process because what should be constant is not the drift rate but the expected return (drift divided by the stock price)!
So for $a = \mu S$ we get

$$dS = \mu S dt + \beta(S, t) dz$$

Assuming that the volatility is zero, $dS = \mu S dt$ or

$$\frac{dS}{S} = \mu dt$$

Integrating between time $0$ and $T$ we get

$$S_T = S_0 e^{\mu T}$$

So if there is no volatility, the stock price grows at a continuously compounded rate of $\mu$ per unit time.

What about $\beta(S,t)$?

$\beta$ should be proportional to the stock price, $\beta \sim S$. (Why?)

By setting $\beta = \sigma S$ we get

$$dS = \mu S dt + \sigma S dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz$$

The variable $\sigma$ is the volatility of the stock price and $\mu$ is its expected return. This is also known as geometric Brownian motion. Notice that

$$\frac{\Delta S}{S} \sim \phi\left(\mu \Delta t, \sigma\sqrt{\Delta t}\right)$$

Example of Geometric Brownian Motion

A stock that pays no dividends has an expected annual return of $15\%$ and annual volatility of $30\%$. In this case, $\mu = 0.15$ and $\sigma = 0.3$. So,

$$\frac{dS}{S} = 0.15dt + 0.3dz$$

or

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.3\epsilon\sqrt{\Delta t}$$

Consider a time interval of one month or $0.08$ years and suppose that the initial price is $100$.

Then,

$$\begin{aligned}\Delta S &= 100(0.012 + 0.085\epsilon)\\ &= 1.2 + 8.5\epsilon\end{aligned}$$

The parameters $\mu$ and $\sigma$

The value of $\mu$ should depend on the risk of the investment as well as the interest rates in the economy.
It turns out that the value of a derivative which depends on a stock is, in general, independent of $\mu$.

On the other hand, $\sigma$ is very important to the determination of the value of many derivatives. There are many ways to estimate the volatility, from very simple ones to very complex models.

Itô’s Lemma

If a variable $x$ follows the Itô process

$$x = \alpha(x,t)dt + \beta(x,t)dz$$

then a function $G = G(x,t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}\alpha + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\beta^2\right)dt + \frac{\partial G}{\partial x}\beta dz$$

## <u>The Lognormal Property of the Stock Prices</u>

In the case of $dS = \mu S dt + \sigma S dz$, using the Itô's lemma for a function $G(S)$ we can write

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

We are interested in the logarithm of the stock price $S$. Hence, we assume $G = \ln S$.
In this case,

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \qquad \text{and} \qquad \frac{\partial G}{\partial t} = 0$$

As a result,

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$$

From the above, $G$ follows a generalized Wiener process with drift rate $\mu - \sigma^2/2$ and variance rate $\sigma^2$.
So between time $0$ and $T$,

$$\Delta G \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

or

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

Since $\ln S_T$ is normally distributed, $S_T$ is lognormally distributed.