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# Boltzmann distribution and market temperature

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**Abstract**

We show that the minute fluctuations of S&P 500 and NASDAQ 100 indices show Boltzmann statistics over a wide range of positive as well as negative returns, thus allowing us to define a *market temperature* for either sign. With increasing time the sharp Boltzmann peak broadens into a Gaussian whose volatility $\sigma$ measured in $1/\sqrt{\text{min}}$ is related to the temperature $T$ by $T = \sigma/\sqrt{2}$. Plots over the years 1990–2006 show that the arrival of the 2000 crash was preceded by an increase in market temperature, suggesting that this increase can be used as a warning signal for crashes.


It is by now well known that financial data do not display Gaussian distributions [1–13]. Most importantly, the tails of the distributions are power like [14], since large fluctuations are much more frequent than in a Gaussian distribution. This is of great importance for financial institutions who want to estimate the risk of market crashes.

In this note we would like to focus on the opposite regime of the most frequent events near the peak of the distribution. The logarithms of the stock prices $x(t) = \log S(t)$ and thus also of NASDAQ 100 and S&P 500 indices have a special property: the minute returns $z(t) = \Delta x(t)$ show an exponential distribution [15] for positive as well as negative $z(t)$, as long as the probability is rather large [16,17].

$$\tilde{B}(z) = \frac{1}{2T} \mathrm{e}^{-|z|/T}. \tag{1}$$

In Fig. 1 we show that the data are fitted well by the distribution [18]. Only a very small set of rare events of large $|z|$ does not follow the exponential law, but displays heavy tails. If the exponential distribution is interpreted as a Boltzmann distribution, the parameter $T$ in (1) plays the role of a market temperature, and there are statistical considerations to support this interpretation [19,20]. The purpose of this note is to determine the market temperatures for the S&P 500 and NASDAQ indices over many years.

In principle, there are different temperature $T_\pm$ for positive and negative returns, but to a good approximations we may equate both $T \approx T_+ \approx T_-$.

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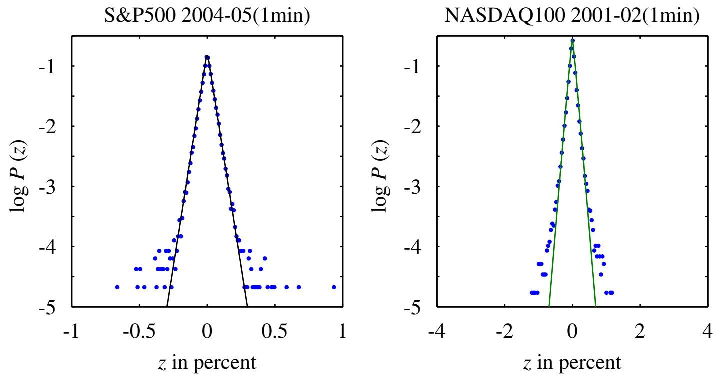


Fig. 1. Boltzmann distribution of minute returns of S&P 500 and NASDAQ 100 indices.

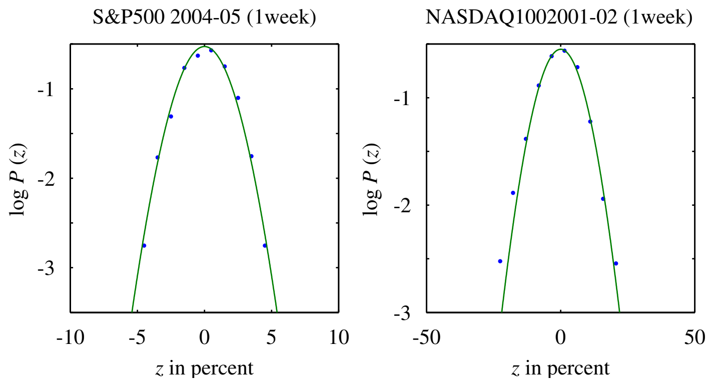


Fig. 2. Gaussian distributions of S&P 500 and NASDAQ 100 weekly returns.

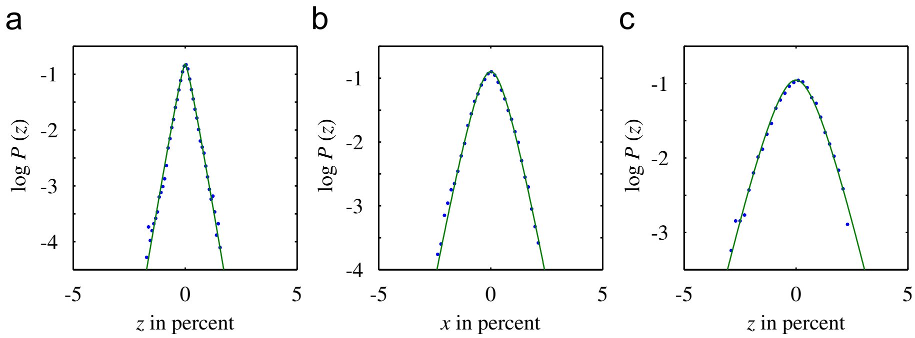


Fig. 3. Fits of convolution of Boltzmann distribution to S&P 500 returns in Fig. 1 over time intervals of 1 h, 4 h, and 1 day, respectively.

At larger time scales, the distribution becomes more and more Gaussian, as required by the *central limiting theorem* of statistical mechanics which states that the convolution of infinitely many arbitrary distribution functions of finite width always approaches a Gaussian distribution. This is illustrated by the weekly data of the two indexes in Fig. 2.

The transition from Boltzmann to Gaussian distributions is shown for the S&P 500 index in Fig. 3.

The convergence to a Gaussian distribution is in contrast to the pure Lévy distribution of infinite width where the falloff remains power-like at large distances for any data frequency. This will happen here as well for the rare events outside of the Boltzmann regime.

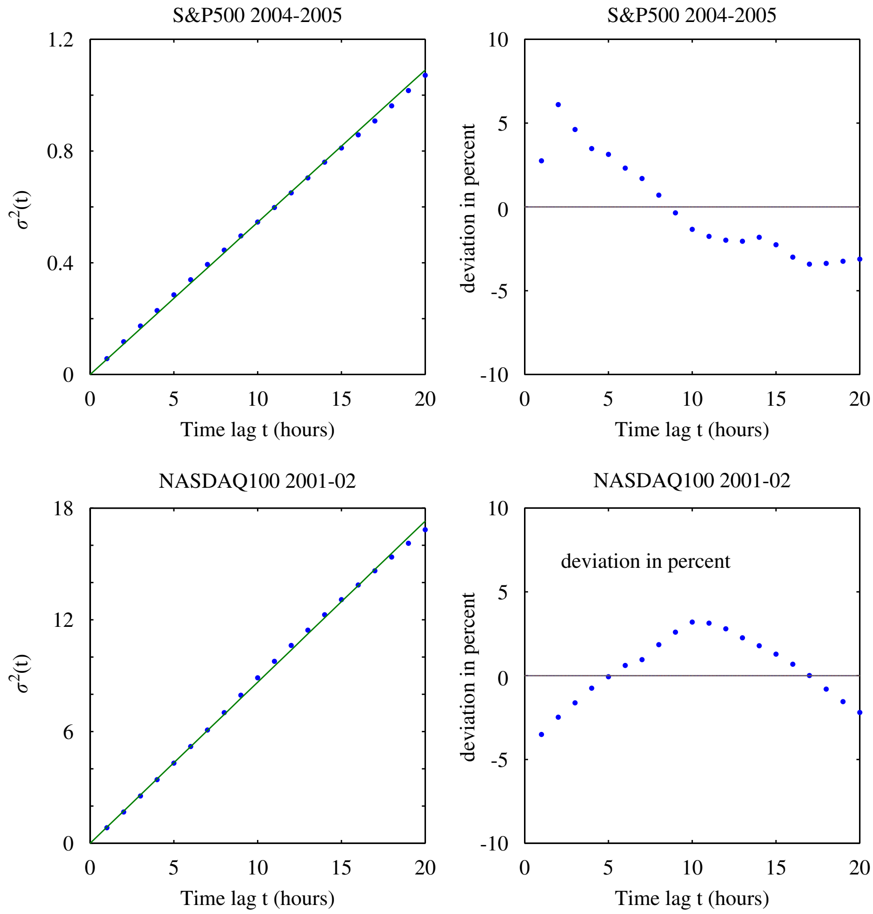


Fig. 4. Variance of S&P 500 and NASDAQ 100 indices as a function of time. The right-hand side amplifies the small relative deviation from the linear shape in percent.

The time dependence of the distribution is found in the usual way [4,13]. We calculate the Fourier transform of $B(z)$:

$$B(p) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}pz} \frac{1}{2T} \mathrm{e}^{-|z|/T} = \frac{1}{1+(Tp)^2}, \tag{2}$$

and identify the Hamiltonian as

$$H(p) = \log[1+(Tp)^2]. \tag{3}$$

This has only even cumulants ($n = 2, 4, \ldots$):

$$c_n = -\mathrm{i}^n H^{(n)}(0) = 2\mathrm{i}^n(-1)^{n/2} T^n (n-1)!. \tag{4}$$

As a function of time, the distribution widens as follows:

$$\begin{aligned}\tilde{B}(z;t) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}pz - tH(p)} \\ &= \frac{1}{T\sqrt{\pi}\Gamma(t)} \left(\frac{|z|}{2T}\right)^{t-1/2} K_{t-1/2}(|z|/T),\end{aligned} \tag{5}$$

where $t$ is measured in minutes. For $t = 1$ this agrees, of course, with the minute distribution (1).

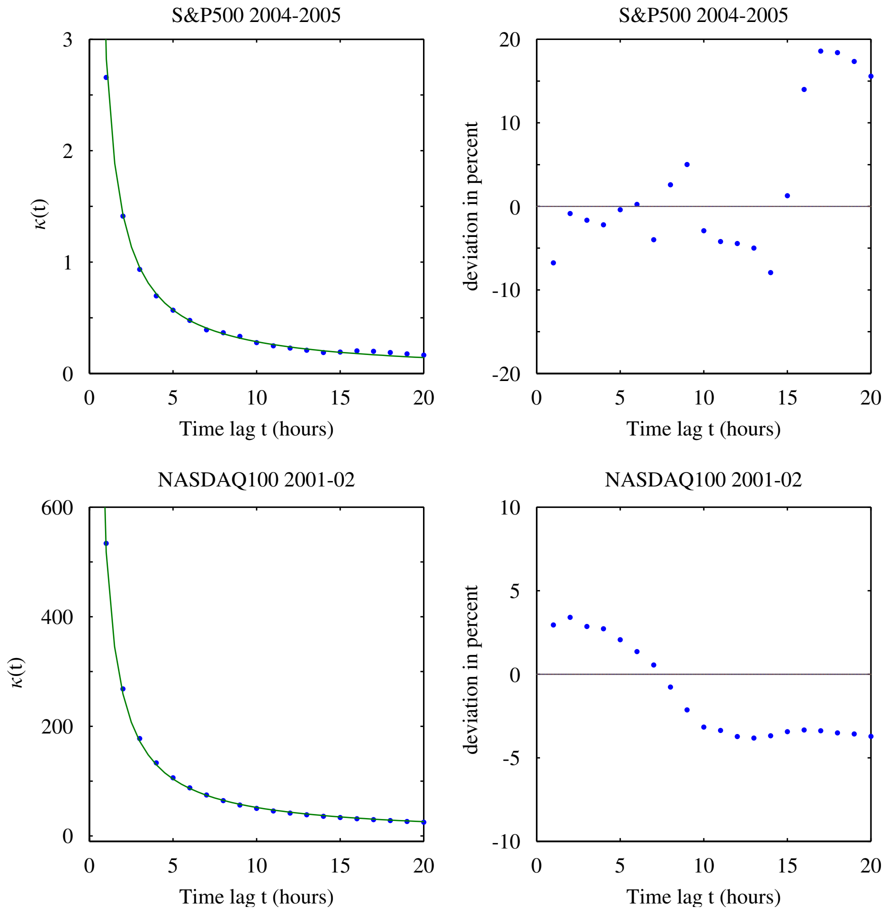


Fig. 5. Kurtosis of S&P 500 and NASDAQ 100 indices as a function of time. The right-hand side shows the relative deviation from the $1/t$ behavior in percent.

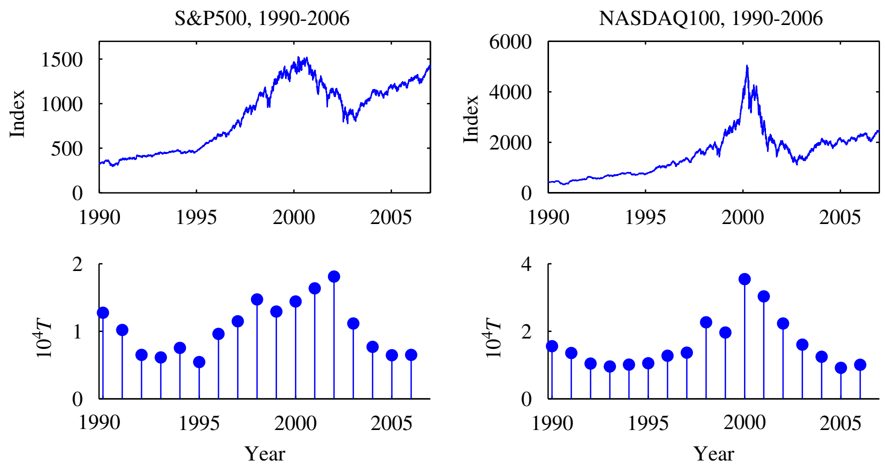


Fig. 6. Market temperatures of S&P 500 and NASDAQ 100 indices from 1990 to 2005. The crash in the year 2000 occurred at the maximal temperatures $T_{\mathrm{S\&P500}} \approx 2 \times 10^{-4}$ and $T_{\mathrm{NASDAQ}} \approx 4 \times 10^{-4}$.

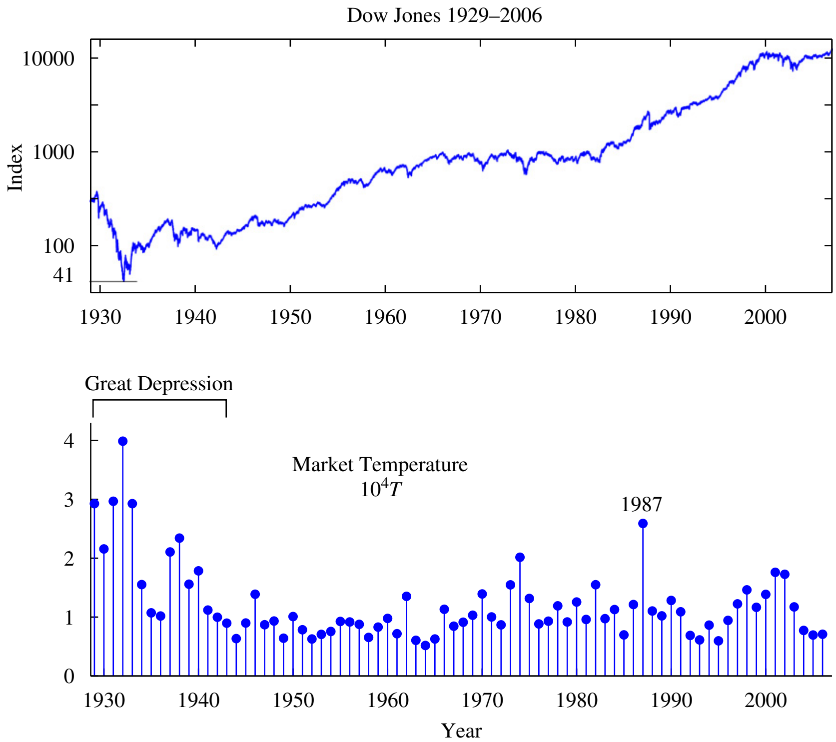


Fig. 7. Dow Jones index over 78 years (1929–2006) and the yearly market temperature, which is remarkably uniform, except in the 1930's, in the beginning of the great depression. Another heat burst occurred in the crash year 1987 (data from Ref. [21]).

The variance of this distribution increases linearly in time as

$$\sigma^2(t) \equiv \langle z^2 \rangle_c(t) = \sigma^2 t = 2T^2 t, \tag{6}$$

whereas the kurtosis decreases with $1/t$

$$\kappa(t) \equiv \frac{\langle z^4 \rangle_c(t)}{\langle z^2 \rangle_c^2(t)} - 3 = \frac{3}{t}, \tag{7}$$

and goes to zero for large times where the distribution becomes Gaussian.

These quantities are plotted in Figs. 4 and 5. The time dependence of $\sigma^2(t)$ in Eq. (6) allows us to extract the temperature of the initial Boltzmann distribution as $T = \sqrt{\sigma^2(t)/2t}$ from any later distribution in which the sharp Boltzmann peak is no longer visible, in particular, from the asymptotic Gaussian limit. The result of this analysis is contained in the plots of Fig. 6. The temperature depends, of course, on the selection of stocks, but changes only very slowly with the general economic and political environment. Near a crash, however, it increases significantly.

It is interesting to observe the historic development of Dow Jones temperature over the last 78 years (1929–2006) in Fig. 7. Although the world went through a lot of turmoil and economic development in the 20th century, the temperature remained rather constant except for short heat bursts. The temperature was highest in the 1930's, the time of the great depression. These temperatures have never been reached again. An especially hot burst occurred during the crash year 1987.

The lesson from this analysis is that an increase in market temperature before a crash may be a useful signal for investors to shorten their positions.

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