Relation between volatility correlations in financial markets and Omori processes occurring on all scales

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We analyze the memory in volatility by studying volatility return intervals, defined as the time between two consecutive fluctuations larger than a given threshold, in time periods following stock market crashes. Such an aftercrash period is characterized by the Omori law, which describes the decay in the rate of aftershocks of a given size with time t by a power law with exponent close to 1. A shock followed by such a power law decay in the rate is here called Omori process. We find self-similar features in the volatility. Specifically, within the aftercrash period there are smaller shocks that themselves constitute Omori processes on smaller scales, similar to the Omori process after the large crash. We call these smaller shocks subcrashes, which are followed by their own aftershocks. We also show that the Omori law holds not only after significant market crashes as shown by Lillo and Mantegna [Phys. Rev. E **68**, 016119 (2003)], but also after "intermediate shocks." By appropriate detrending we remove the influence of the crashes and subcrashes from the data, and find that this procedure significantly reduces the memory in the records. Moreover, when studying long-term correlated fractional Brownian motion and autoregressive fractionally integrated moving average artificial models for volatilities, we find Omori-type behavior after high volatilities. Thus, our results support the hypothesis that the memory in the volatility is related to the Omori processes present on different time scales.

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INTRODUCTION

The correlations of stock returns are important for risk estimation, and can be used for forecasting financial time series. The absolute value of the return, which is a measure for volatility, seems to have a memory [1–17], so that a return is more likely to be followed by a return with similar absolute value, which leads to periods of large volatility and other periods of small volatility (called volatility clustering in economics). While the absolute value exhibits long-term correlations decaying as a power law [18], the correlations of the return itself decay exponentially with a characteristic time scale of 4 min [13,16].

Recent studies [19–22] reveal more information about the temporal structure of the volatility time series by analyzing volatility return intervals, the time between two consecutive events with volatilities larger than a given threshold. These return intervals display memory and volatility clustering, and also scaling properties for different thresholds, which seem to be universal for different time scales and markets [19–22]. This behavior is similar to what is found in earthquakes [23] and climate [24,25]. Rare extreme events such as market crashes constitute a substantial risk for investors, but these rare events do not provide enough data for reliable statistical analysis. Due to the scaling properties, it is possible to analyze the statistics of return intervals for different thresholds by studying only the behavior of small fluctuations occurring very frequently, which have good statistics.

Lillo and Mantegna studied exclusively three huge stock market crashes and found that after such a market crash the rate of volatilities larger than a given threshold q decreases as a power law with an exponent close to 1 [26]. This behavior is analogous to the classic Omori law describing the aftershocks following a large earthquake [27].

Here, we show that the Omori law holds not only after significant market crashes, but also after "intermediate shocks." Moreover, we find self-similar features in the volatility. Specifically, within the aftercrash period (characterized by the Omori law) there are smaller shocks that themselves behave similar to the Omori law on smaller scales. We call these shocks subcrashes, which can be considered as "new crashes on a smaller scale," followed by their own aftershocks.

Furthermore, we analyze the memory in volatility return intervals after large market crashes, and show that the memory is related to the Omori law. Indeed, if we perform appropriate detrending, the return intervals show significantly less memory, but some memory still exists, independent of the large market crash. We also show that at least part of this "remaining memory" can be described by the self-similar subcrashes: if we also remove Omori processes due to subcrashes, the memory is further reduced. However, some memory still remains so that these crashes cannot account for the entire memory, raising the possibility that the "remaining memory" is due to other subcrashes whose influence was not removed. Moreover, when studying long-term correlated fractional Brownian motion (fBm) and autoregressive fractionally integrated moving average (ARFIMA) artificial models for volatilities, we find Omori-type behavior after high volatilities. Thus, our results support the hypothesis that the memory in the volatility is related to the Omori processes present on different time scales.

This paper is organized as follows. Section I presents information about the analyzed data. In Sec. II we show and discuss the mechanism based on Omori processes on different scales. In Sec. III we study the memory in return intervals induced by large and intermediate shocks and look at Omori processes in long-term correlated artificial models. In Sec. IV we analyze the influence of crashes on the volatility memory, and Sec. V presents discussion and conclusions.

I. THE DATA SETS ANALYZED

In order to capture a variety of market crashes, we analyze three different data sets.

(i) We study the 1 min return time series of the S&P500 index from 1984 to 1989. We analyze the aftercrash period in the 15 000 trading minutes (approximately two months) after "Black Monday," 19 October 1987, as well as after a smaller crash on 11 September 1986. We also analyze the time after several other smaller market crashes within the entire data set.

(ii) The second data set consists of the Trades and Quotes (TAQ) data base of the year 1997 which is provided by the NYSE and contains all trades and quotes for all stocks traded at NYSE, NASDAQ, and AMEX. We choose the 100 most frequently traded stocks and calculate an index by a summation of the normalized prices of each stock (normalized by the first price of the respective time series). From this index, we calculate a 1 min return time series for our analysis, which we analyze in the approximately two months after the crash on 27 October 1997.

(iii) As an example of a crash that is clearly due to an external event, we also study the 1 min return series of General Electric (GE) stock in the three months after 11 September 2001.

For all three data sets, we calculate the volatility as the absolute value of the 1 min return, normalized by the standard deviation σ of the entire period. Hence, in this paper the volatility and also the threshold q are measured in units of the standard deviation σ.

II. OMORI LAW ON DIFFERENT SCALES

Lillo and Mantegna [26] showed that the Omori law [27] for earthquakes also holds after crashes of very large magnitude in financial markets, so that the rate $n(t)$ of events with volatility larger than a given threshold q decays as a power law

$$n(t) = kt^{-\Omega}, \qquad (1)$$

where Ω is around 1 for large q and k is a parameter characterizing the amplitude of the rate $n(t)$. For estimating the parameter k and the exponent Ω, we use the cumulative number $N(t)$ of events larger than q, given by

$$N(t) = \int_0^t n(t')dt' = k\frac{1}{1-\Omega}t^{1-\Omega}. \qquad (2)$$

We study the Omori law on different time scales. Figure 1 shows the cumulative rate $N(t)$ above (a) q=3 and (b), (c) q=4 compared to the volatility in time periods following three significant market crashes in (a) 1986, (b) 1987, and (c) 1997. The volatility is smoothed by a moving average over 60 min in order to remove insignificant fluctuations. The

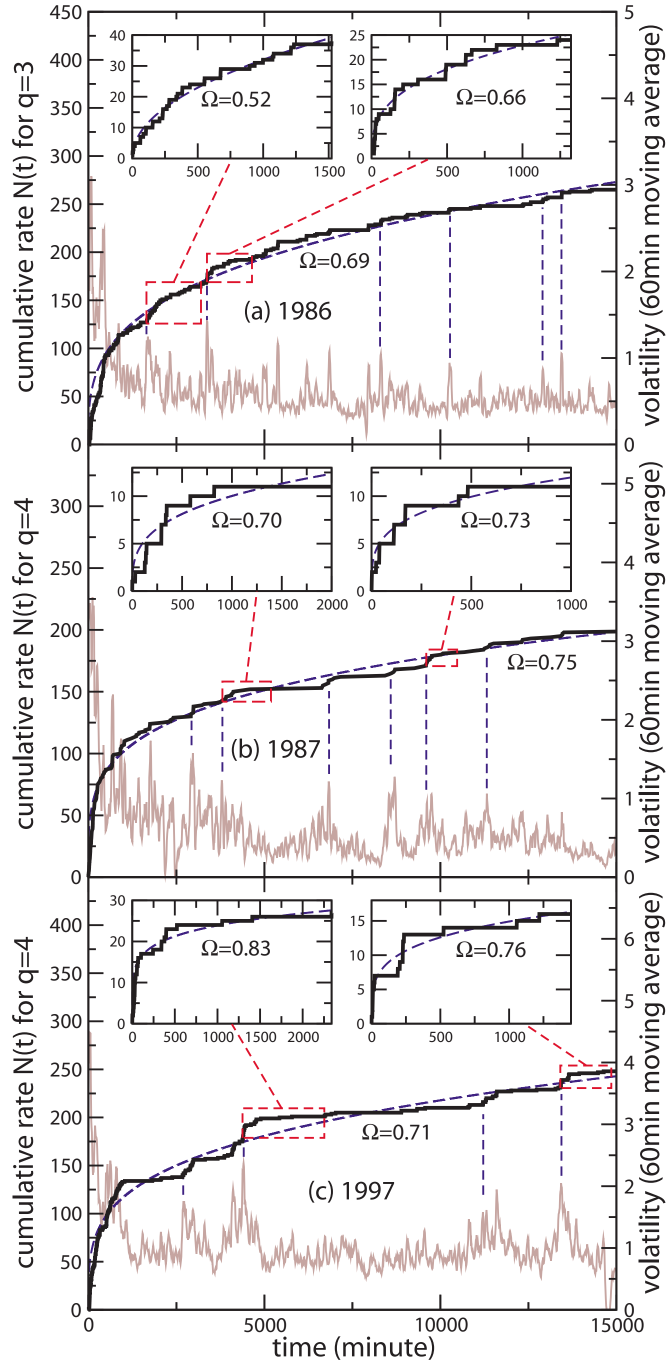

FIG. 1. (Color online) Comparison between volatility and the cumulative rate $N(t)$ of volatilities (absolute 1 min returns) larger than a threshold q. The plots show the 15 000 min (approximately two months) after the market crashes on (a) 11 September 1986, with q=3, (b) 19 October 1987, with q=4, and (c) 27 October 27 1997, with q=4. In each plot (large plots and insets), the empirically found cumulative rate $N(t)$ is represented by the black solid line, whereas the dashed line shows a power law fit to the data in the respective plot. The volatility (gray solid line) is displayed as a moving average over 60 min in order to suppress insignificant fluctuations. The insets show the self-similarity of the data set meaning that while the big crash in the beginning introduces a behavior following the Omori law, some of the aftershocks introduce again a similar behavior on a smaller scale.

large shock in the beginning of the time interval is followed by aftershocks, which induces an Omori-like behavior of $N(t)$ (Omori process), shown by the dashed lines representing a power law fit. However, as seen in Fig. 1 (see insets) many of these aftershocks seem to behave similar to "real" crashes with their own aftershocks (subcrashes), but on a smaller scale (shown by vertical lines). The insets show that a closer look into many of these subcrashes reveals a similar pattern as the Omori law on large scales. The exponent Ω is often smaller after smaller crashes, which is analogous to the finding that the power law decay of the volatility values after smaller shocks has a smaller exponent than after large crashes [28]. Below we explore the possibility that the self-similarity of the volatility (where the Omori law is present on different scales) is directly related to the memory.

III. RETURN INTERVAL MEMORY AFTER CRASHES AND SUBCRASHES

In order to explore the memory effects of the Omori law, we first analyze time periods after very large market crashes. Specifically, we study the memory in the volatility return intervals, which form a sequence of time intervals $\tau(t)$ between two consecutive events with volatilities larger than a given threshold q [19–22]. We next show that the influence of the Omori law on $\tau(t)$ can be estimated by comparing the original $\tau(t)$ with a detrended time series $\tilde{\tau}(t)$ which is independent of the market crash. We fit the cumulative rate $N(t)$ in the period after a market crash with a power law according to Eq. (2), thus obtaining the parameter k and the exponent Ω for the rate $n(t)$ [26]. Using $n(t)$, we can detrend the return interval time series $\tau(t)$ by rescaling by $n(t)$ [29]

$$\tilde{\tau}(t) = \tau(t)n(t). \tag{3}$$

The rational for this detrending is the following: due to the Omori law, Eq. (1), immediately after the crash we have a large rate $n(t)$ of high volatilities so that the return intervals $\tau(t)$ are very short. Later, the rate of high volatilities becomes small while the return intervals get large. This induces memory in the return interval time series since in the beginning small return intervals are followed by more small return intervals, while later large return intervals follow large return intervals. After rescaling according to Eq. (3), high (low) rates and small (large) return intervals cancel each other so that $\tilde{\tau}(t)$ is detrended and thus independent of the existence of the crash, since the trend caused by the crash is no longer present.

The relation between the Omori law and the short-term memory in the return interval time series can be studied by analyzing the conditional expectation value $\langle \tau(t) | \tau_0 \rangle$ of the return interval series $\tau(t)$ conditioned on the previous return interval τ_0 [19,20], for both the original return intervals $\tau(t)$ and the detrended time series $\tilde{\tau}(t)$. In Fig. 2 (left column), $\langle \tau(t) | \tau_0 \rangle$ is plotted against τ_0. Both quantities are normalized by the average return interval $\langle \tau \rangle$, for return intervals after the crashes in (a) October 1987 and (b) October 1997. The deviations from a horizontal line at 1 for all thresholds show memory: large (small) values of τ_0 are more likely to be followed by large (small) values of $\tau(t)$. The slopes of the curves for the detrended time series $\tilde{\tau}$ are significantly less steep (right column), indicating that detrending the Omori law from the time series significantly reduces the memory, but some of the memory still remains, which might be due to the Omori process still present on smaller scales (see Fig. 1).

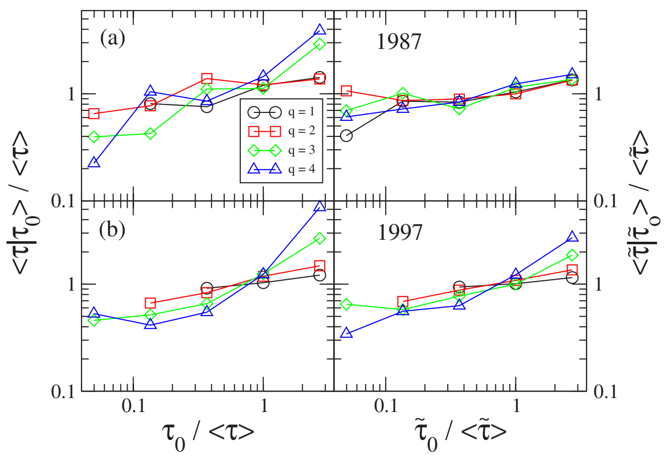

FIG. 2. (Color online) Memory in volatility return intervals for different thresholds before (left column) and after (right column) detrending the time series according to Eq. (3). The analysis is shown for (a) the S&P500 index in the two months after the crash on 19 October 1987 and (b) an index calculated from the 100 most frequently traded stocks from the TAQ data base after the crash of 27 October 1997. Removing the Omori law reduces the memory in the data sets, but some memory still exists.

In addition to the effect of the major crash, we can also analyze the influence of Omori processes after subcrashes on smaller scales. To this end, we further detrend the time series by removing some subcrashes and test whether the memory is further reduced. After identifying the subcrashes [30], we detrend the return intervals $\tau(t)$ by removing the Omori process due to the major crash as well as the Omori processes induced by the subcrashes. To this end, we estimate the parameters k and Ω in Eq. (1) for the rate $n(t)$ after the major crash as well as for the rate $n_s(t)$ in the 1000 min following each subcrash (or the time to the next subcrash, if smaller). Note that $n_s(t)$ is calculated from the detrended return intervals $\tilde{\tau}(t)$. Then, the double detrended return interval time series is given by

$$\tilde{\tilde{\tau}}(t) = \begin{cases} n_s(t)\tilde{\tau}(t) & \text{in time following a subcrash} \\ \tilde{\tau}(t) & \text{otherwise}. \end{cases} \tag{4}$$

In order to improve the statistics for testing the effect of removing also subcrashes on the memory, we plot in Fig. 3 the conditional expectation value $\langle \tau | \tau_0 \rangle / \langle \tau \rangle$ for only two τ_0 intervals: τ_0 below and τ_0 above the median of τ. We see in Fig. 3 that when τ_0 is below the median $\langle \tau | \tau_0 \rangle / \langle \tau \rangle < 1$, while $\langle \tau | \tau_0 \rangle / \langle \tau \rangle > 1$ for τ_0 above the median. This indicates the memory in the records, and also shows that the memory in the original records (circles) gradually weakens upon detrending the time series by removing the influence of the major crash (squares) and further weakens when also some

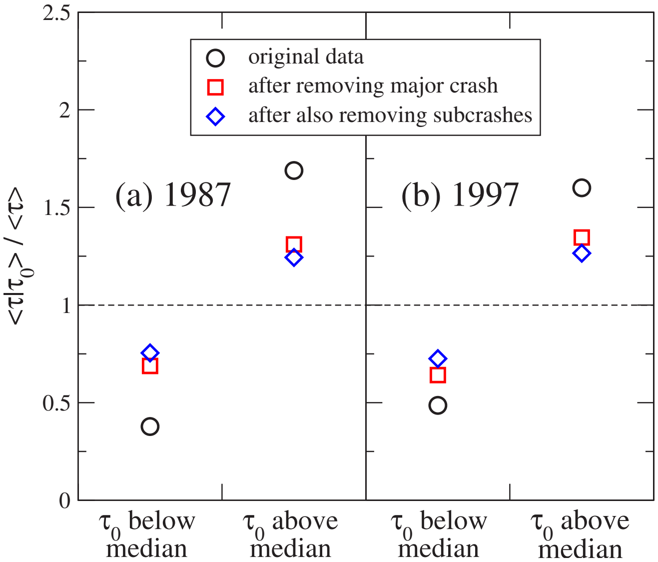

FIG. 3. (Color online) Memory in volatility return intervals for threshold $q=3$ for (a) the S&P500 index in the two months after the crash on 19 October 1987 and (b) for an index calculated from the 100 most frequently traded stocks from the TAQ data base after the crash of 27 October 1997. The conditional expectation value $\langle\tau|\tau_0\rangle/\langle\tau\rangle$ conditioned on the previous return interval τ_0 is smaller than 1 if τ_0 is below the median while $\langle\tau|\tau_0\rangle/\langle\tau\rangle>1$ if τ_0 is above the median, indicating the memory in the records (circles). The effect gradually weakens upon detrending the time series by removing the influence of the major crash (squares) and even further when removing some subcrashes (diamonds).

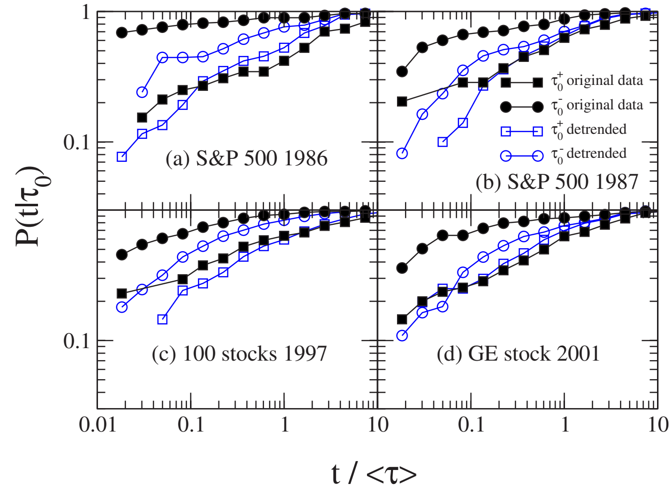

FIG. 4. (Color online) Probability $P(t|\tau_0)$ that after a return interval τ_0 the next volatility larger than a threshold $q=4$ [$q=3$ in (d)] occurs within time t. Here, τ_0 belongs to either the 25% smallest values (τ_0^-, circles) or the 25% largest values (τ_0^+, squares) of τ. The memory in the original time series (filled symbols) is reduced by detrending according to Eq. (3) (open symbols), but some of the memory still remains. The results are shown for (a) the S&P500 index after a crash on 11 September 1986, (b) the S&P500 index after the crash on 19 October 1987, (c) an index created from the 100 most frequently traded stocks from the TAQ database after the crash on 27 October 1997, and (d) General Electric (GE) stock after 11 September 2001.

subcrashes are removed (diamonds). Hence, not only a large market crash but also smaller subcrashes contribute to the memory in return intervals.

To further investigate the effect of removing the memory induced by aftershocks, we analyze the probability $P(t)$ that after an event larger than a certain volatility q the next volatility larger than q appears within a time t [21,23,25]. In order to study the effect of memory, we plot the conditional probability $P(t|\tau_0)$ for different values of the preceding return interval τ_0. Figure 4 shows $P(t|\tau_0)$ for $q=2$ under the condition that the preceding return interval τ_0^- belongs to the smallest 25% of the return intervals or that the preceding return interval τ_0^+ belongs to the largest 25%. The memory in the time series leads to a splitting of the curves because after larger return intervals (squares) the time to the next volatility above q is usually large, while it is short after small return intervals (circles). After detrending the time series the two curves get closer, indicating a reduced memory, but also here some memory still remains.

To test the long-term memory effects of the Omori process on the volatility return intervals we study the autocorrelation function shown in Fig. 5 for return intervals after the market crashes in 1987 and 1997 for two different thresholds $q=1$ and $q=2$. For both thresholds, we see that there exists a significant correlation even between return intervals 100 steps apart, which corresponds to approximately 2 to 5 days in 1987 (0.5 to 2 days in 1997) since the average return intervals are $\langle\tau(q=1)\rangle=6.33$ min and $\langle\tau(q=2)\rangle=17.4$ min in 1987 and $\langle\tau(q=1)\rangle=2.47$ min and $\langle\tau(q=2)\rangle=7.66$ min in 1997. If we now remove the effect of the Omori process due to the market crash by detrending according to Eq. (3), the memory in the detrended sequence $\tilde{\tau}$ is reduced significantly, as we see in the dashed curves of Fig. 5. The dotted lines show that removing also the influence of some subcrashes according to Eq. (4) further reduces the memory.

So far, we showed indications that within the time period after a big crash there might exist smaller crashes that behave similar to the big crash. The question arises whether such subcrashes are only typical after a large crash or whether they appear in all time periods independent of the existence of a big crash. To test this, we analyze if Omori processes exist also for smaller crashes. We study 22 crashes of sizes between 11 and 16 standard deviations in the S&P500 time series from 1984 to 1989. These crashes are considerably smaller than the huge crashes of more than 30 standard deviations in a 1 min interval studied above. We analyze the cumulative rate $N(t)$ in the 1000 trading minutes following these smaller crashes. In order to make different crashes comparable irrespective of the current trading activity, we normalize the cumulative rate $N(t)$ by $N(1000)$. Figure 6(a) shows this normalized rate $N(t)/N(1000)$ averaged over all aftershock periods [31]. For different thresholds q, $N(t)/N(1000)$ can be fit with a power law (2). The exponent Ω increases with the threshold, but is generally smaller than the exponents found after very large shocks. Our results for the *rate* decay are analogous to volatility studies [28,32] where the exponent characterizing the *volatility* decay depends on the magnitude of the shock [28]. These results indicate that relatively small crashes have similar Omori processes which may lead to memory effects.

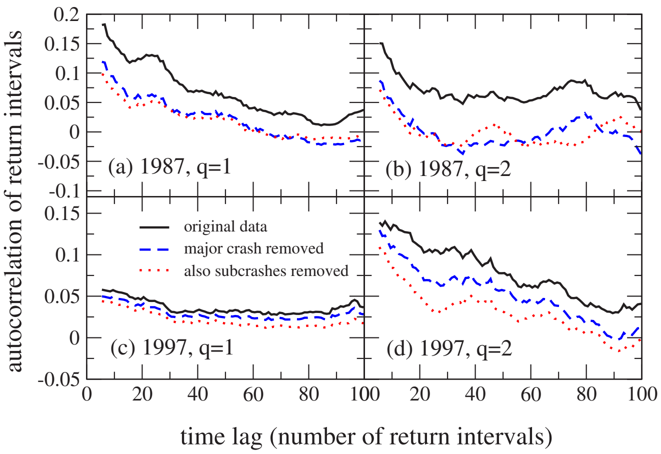

FIG. 5. (Color online) Autocorrelation function of the return interval time series for threshold (a),(c) $q=1$ and (b),(d) $q=2$. The first row (a),(b) shows results from the S&P500 index in the three months after the market crash on 19 October 1987, while the second row (c),(d) results from an index created from the 100 most frequently traded stocks from the TAQ database after the crash on 27 October 1997. The Omori law due to the market crash (original data, solid lines) induces correlations leading to an offset in the autocorrelation function which is removed in the detrended $\tilde{\tau}$ (dashed lines), but the data still show some long-term correlations even after removing the influence of the Omori law. However, after further detrending with respect to some subcrashes (dotted line), the autocorrelation is further reduced. All lines are smoothed by a moving average over ten return intervals.

In order to test how generic the relation between the described Omori processes and the memory in volatility is, we analyze artificial time series with power law autocorrelations. To this end, we simulate a common model for volatilities, the autoregressive fractionally integrated moving average (ARFIMA) process [33,34], where the time series of price changes $\{\delta g_i\}$ is given by

$$\delta g_i^{\mathrm{ARFIMA}} = \sum_{n=1}^{\infty} a_n(\rho)\,\delta g_{i-n} + \eta_i, \tag{5}$$

$$a_n(\rho) = \rho \frac{\Gamma(n-\rho)}{\Gamma(1-\rho)\Gamma(1+n)}. \tag{6}$$

Here, the variables η_i are normal distributed random numbers with mean 0 and variance 1. While the parameter ρ determines the autocorrelations of δg_i, the autocorrelation function of $|\delta g_i|$ is independent of ρ, following a power law with exponent -0.5. We adjust the distribution of the generated data so that it is Gaussian for $|\delta g_i| \leq 2$, but matches the empirical data with a power law distribution with exponent 4 for $|\delta g_i| > 2$ [35]. Due to this procedure the autocorrelation function changes as well, resulting in power law autocorrelations with exponent 0.19 that are similar to values found empirically (e.g., in Ref. [32]).

In addition to the ARFIMA process, we also simulate a fractional Brownian motion (fBm) with

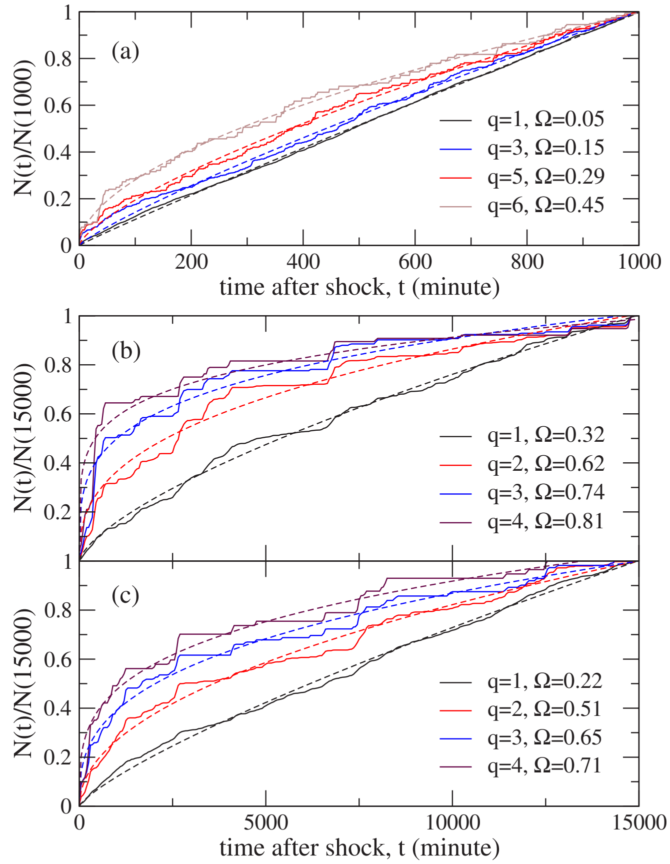

FIG. 6. (Color online) (a) Cumulative rate $N(t)$ of events larger than a threshold q averaged over the 1000 min after 22 shocks between 11σ and 16σ in the S&P500 one minute time series of the years 1984 to 1989. The data for each shock is normalized by $N(1000)$ in order to make different shocks comparable irrespective of the current trading activity. Each cumulative rate (solid lines) for different thresholds q can be well fitted by a power law (dashed line) according to Eq. (2). The curves are displayed for $q=1,3,5,6$, where the exponent grows from $\Omega=0.05$ to $\Omega=0.45$. Hence, the bottom curve corresponds to $q=1$ with the smallest Ω whereas the top curve represents $q=6$ with the largest Ω. (b) Omori law (solid lines) after a large shock in the simulation of an ARFIMA model with power law autocorrelations (exponent 0.19) and a cumulative distribution with power law tails (exponent 3). The exponent obtained for the Omori law by a power law fit, Eq. (2), (dashed line) ranges from $\Omega=0.32$ to $\Omega=0.81$ for $q=1,\ldots,4$ (bottom curve: $q=1$, top curve: $q=4$). (c) In the simulation of a fractional Brownian motion (fBm) with power law autocorrelations (exponent 0.36), a large shock is also followed by an Omori process (solid line). Here, the exponent ranges from $\Omega=0.22$ to $\Omega=0.71$ for $q=1,\ldots,4$ (bottom curve: $q=1$, top curve: $q=4$).

$$\delta g_i^{\mathrm{fBm}} = G(t)e^{\chi(t)}, \tag{7}$$

where $\chi(t)$ is a fractional Gaussian noise and $G(t)$ a Gaussian noise. The volatility is obtained from the absolute value $|\delta g_i|$, which, in our simulation, exhibits power law autocorrelations with exponent 0.36.

Figures 6(b) and 6(c) show the normalized cumulative rate $N(t)$ for the 15 000 time steps after a large price change in the generated data from (b) the ARFIMA process and (c)

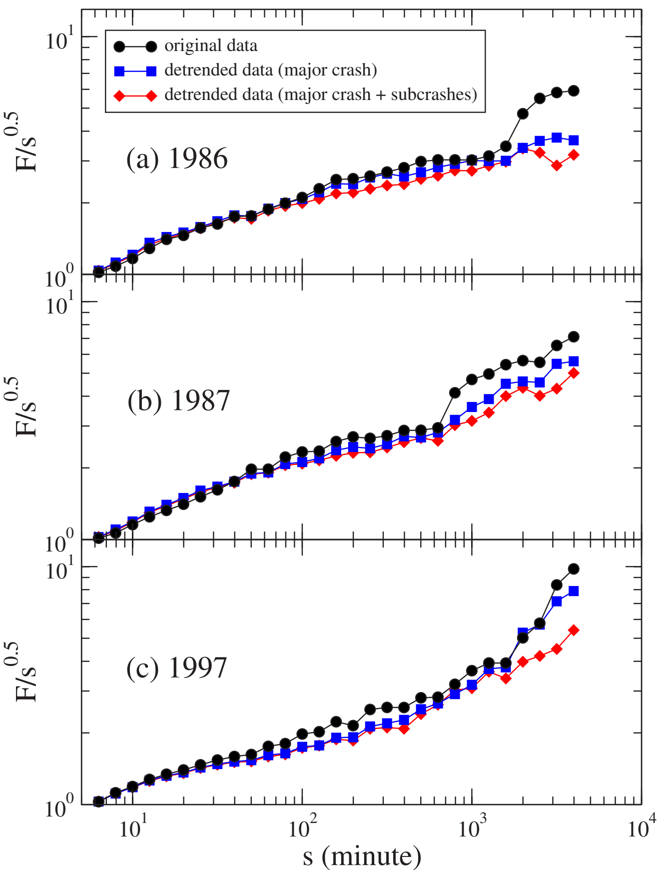

FIG. 7. (Color online) Root mean square fluctuation $F(s)$ obtained by the second order DFA method (DFA2) for the volatility in the 15 000 min following market crashes in (a) the S&P500 index on 11 September 1986 and (b) on 19 October 1987, as well as (c) the market crash on 27 October 1997 for an index created from TAQ data for 100 stocks. $F(s)$ is divided by $s^{0.5}$ to clarify the deviation from uncorrelated data. Compared to the original volatility $v(t)$ (circles), the memory is reduced in the detrended records $\tilde{v}(t)$ (squares), and even further after also detrending some subcrashes in $\tilde{\tilde{v}}(t)$ (diamonds).

the fBm. The curves for different thresholds $q=1,\ldots,4$ indicate that also in the simulations a large crash initiates an Omori process with an increasing exponent for larger thresholds q. In addition, there seem to be Omori processes on smaller scales as well. These results indicate that there is a strong relation between the power law autocorrelations found in the volatility and the occurrence of Omori processes, which has been found by Lillo and Mantegna for major market crashes [26]. Omori-type laws appear also in the multi-time-scale model recently presented by Borland and Bouchaud [36], which can also account for volatility clustering.

IV. MEMORY IN VOLATILITY AFTER CRASHES AND SUBCRASHES

In the previous sections, we showed that the memory in return intervals decreases when we remove effects due to Omori processes. Since the studied return intervals $\tau(t)$ are

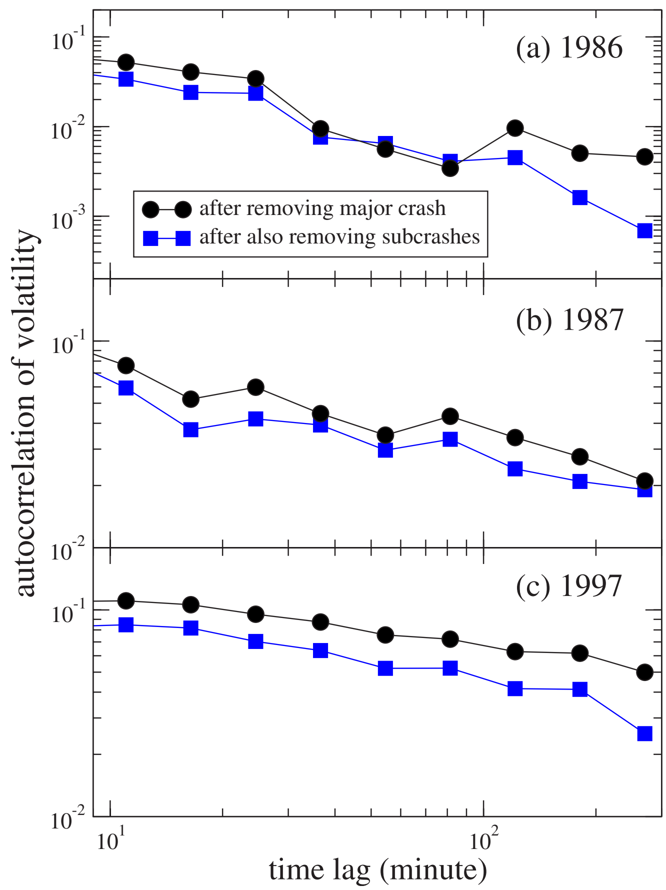

FIG. 8. (Color online) Autocorrelation function of the volatility time series after detrending. Compared to the volatility time series after detrending the major crash (circles), detrending subcrashes (squares) further reduces the autocorrelations. The results are shown for (a) the S&P500 index after a crash on 11 September 1986, (b) the S&P500 index after the crash on 19 October 1987, (c) an index created from the 100 most frequently traded stocks from the TAQ database after the crash on 27 October 1997. The autocorrelation function of the original volatility time series is not shown because it is not meaningful as it is dominated by the influence of the market crash.

derived from the volatility time series $v(t)$, it would be interesting to test whether the memory in $v(t)$ is also affected by Omori processes. Thus, we next analyze the effect of Omori processes on the memory in the volatility time series directly. It is known that a market crash induces a power law decay of the approximate form

$$v_{\mathrm{PL}}(t) \equiv v_0 t^{-\beta} \tag{8}$$

with an exponent $\beta \approx 0.2-0.3$ [26,28]. In order to study the memory induced by this decay, we compare the original time series $v(t)$ to a detrended one

$$\tilde{v}(t) \equiv \frac{v(t)}{v_{\mathrm{PL}}(t)} \tag{9}$$

so that $\tilde{v}(t)$ does not depend on the market crash.

We use second order detrended fluctuation analysis (DFA2) [37,38] to study the long-term memory in the volatility [1–17]. In DFA2, the fluctuations $F(s)$ (root mean square fluctuations) from a second degree polynomial fit of the profile

$$y(t) = \sum_{t'=0}^{t} v(t') \tag{10}$$

as a function of different scales s (time windows) reveal information about the memory. If $F(s) \sim s^{\alpha}$, the autocorrelation exponent γ of the time series is related to the exponent α by $\alpha = 1 - \gamma/2$. For $\alpha > 0.5$, the time series is long-range correlated, it is anticorrelated for $\alpha < 0.5$, and $\alpha = 0.5$ indicates no long-range correlations. Figure 7 shows $F(s)/s^{0.5}$ plotted against s in a log-log plot for 15 000 trading minutes after three different market crashes of 1986, 1987, and 1997. With no long-term correlations, the function would be constant, while a positive slope indicates long-term correlations. For all crashes, the original time series (circles) shows an increased slope on large time scales. After detrending according to Eq. (9) and replacing $v(t')$ by $\tilde{v}(t')$ in Eq. (10), the curve (squares) gets less steep, indicating a reduction of the memory (the curves are shifted so that they start at the same point).

As described before, there are also subcrashes that may induce their own power law decay on a smaller scale—not only in the rate, but also in the volatility values. In order to analyze the memory due to these subcrashes, we further detrend the time series and test whether the memory is reduced further. To this end, we fit the detrended volatility $\tilde{v}(t)$ in the 1000 min following each subcrash (or the time to the next subcrash, if shorter) with a power law $\tilde{v}_{\mathrm{PL}}$ according to Eq. (8). Then, we further detrend $\tilde{v}(t)$ in these regions using Eq. (9) for $\tilde{v}(t)$ instead of $v(t)$. The DFA2 curve for the double detrended time series $\tilde{\tilde{v}}(t) \equiv \tilde{v}/\tilde{v}_{\mathrm{PL}}$ is shown in Fig. 7. The decrease in the slope shows that the memory is further reduced after removing the influence of the subcrashes. However, we clearly see that removing the trends induced by a market crash as well as by subcrashes slightly reduces the memory in the volatility on quite small scales ($s < 60$ min).

The effect of removing subcrashes on the long-term correlations of volatility is seen better in Fig. 8. Here, we compare the autocorrelation functions of the detrended volatility $\tilde{v}(t)$ and the double detrended volatility $\tilde{\tilde{v}}(t)$ after also removing subcrashes. It is seen that generally the autocorrelation of $\tilde{\tilde{v}}(t)$ is smaller compared to $\tilde{v}(t)$, which indicates that the Omori processes after subcrashes also contain some memory.

V. DISCUSSION AND CONCLUSIONS

We find that Omori processes after market crashes exist not only on very large scales, but a similar behavior is also induced by less significant shocks. Moreover, we show that such Omori processes on different scales can occur within the same time period. This leads to self-similar features of the volatility time series, meaning that some of the aftershocks of a large crash can be considered as subcrashes that themselves initiate Omori processes on a smaller scale.

We ask the question whether this self-similarity can be responsible for the memory in volatility return intervals as well as for the memory of the volatility itself. Our results show that a significant amount of memory is induced by these crashes and subcrashes, which suggests that at least a large part of the memory in volatility might be due to Omori processes on different scales. We also show that artificial long-term correlated data exhibit behavior similar to the Omori law. Thus, we believe that there is a strong relation between Omori processes and the long-term correlation found for volatility sequences of financial markets.

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correlation function of absolute returns. Later, attempts were made to quantify this slow decay. For example. Ding *et al.* [9] analyzed daily returns of the S&P500 index time series for a period of more than 60 years. They found that a power law fit of the autocorrelation function of the absolute return decreases too fast in the beginning (i.e., short time lags) but too slow for long time lags. Hence, they fit the data with a combination of an exponential function and a power law. Dacorogna *et al.* [10] studied the autocorrelation of the absolute return in the foreign exchange market. Using four years of 20 min returns of different exchange rates, they find that a hyperbolic curve (i.e., a power law) fits the data much better than an exponential curve. The power law exponent varies between 0.2 and 0.3 depending on the exchange rate. Moreover, they found that the decay becomes faster when considering very large time lags of more than 10 days. Liu *et al.* [13,16] analyzed the 1 min returns of the S&P500 index over a 13 year-period and found that the autocorrelation of the absolute return exhibits a power law decay with exponent 0.3.

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