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Majority-vote model for financial markets

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HIGHLIGHTS

- A generalized majority-vote model is used to model financial markets.
- Two agents: fundamentalists and noise traders with different couplings.
- The model presents stylized facts of real financial markets.

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ABSTRACT

We use a heterogeneous agent-based two-state sociophysics model to simulate financial markets. Focusing on stock market trader dynamics, we propose a model with two kinds of individual – the contrarian agent and the noise trader – in which the dynamics of buying and selling investors are governed by local and global interactions. We define an antiferromagnetic coupling that relates the option of contrarian agents to global magnetization and a ferromagnetic interaction that connects noise traders to their local neighborhood. Our model presents such stylized facts of real financial markets as clustered volatility, power-law distributed returns, and the long-time correlation of the absolute returns with exponential decay. We also observe that the distribution of logarithmic returns can be fitted by the Student's t distribution in which its degree of freedom changes with the percentage of contrarian agents in the market.

1. Introduction

In recent years there has been widespread interest in using the techniques and methods of statistical mechanics to investigate the trader dynamics of financial markets and the spread of opinions in a social network. This use of the mathematical tools and methods of physics to understand the dynamics of financial markets and human interactions has produced the interdisciplinary fields of econophysics and sociophysics, respectively, and many agent-based models have proposed to describe the dynamics of traders in a financial market or opinions in a social network [1–16].

Using the Ising model, an agent-based model was proposed to investigate the factors that drive expectation dynamics in financial and stock markets [4]. The model introduces a coupling that relates each spin to the global magnetization of the system and to the Ising couplings connecting each spin to its nearest neighbors. For temperatures below the critical temperature of the Ising model, the competition between local and global interactions causes the magnetization dynamics

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of the system to become metastable. Despite its simplicity this model presents such main features observed in real markets as volatility clustering and long-time correlations with exponential decay, and thus allows us to understand the basic mechanisms at work in real-world financial markets [4–6].

The majority-vote model is another widely studied agent-based statistical physics model [9,10]. It shows the opinion dynamics in a society by assuming that each spin in a network of interactions is either $+1$ or -1, a value that represents the opinion of each network agent. Each spin in the model assumes with probability q a state opposite to that of the majority of its neighboring spins and with probability $(1-q)$ the same state. Here q is the noise parameter of the model and measures system temperature. As in the Ising model, the majority-vote model exhibits a phase transition at a critical value of the noise parameter, which is $q_c \approx 0.075$ in a square lattice network. We find that the majority-vote model has the same critical exponents as the Ising model [11–14].

Financial markets are driven by rational and emotional behavior of their agents, which first appears to be difficult to simulate and predict. Besides that, there are few situations where the human behavior tends to be simple to anticipate and may be well fitted using spin models. In social systems, humans tend to adopt herding behavior and follow the crowd because they feel more comfortable when their decisions are supported by the like decisions of other people. In financial markets this behavior is reinforced by so-called noise traders agents, who follow trends and over-react to both good and bad news when they buy and sell. In contrast, some agents find that following the global *minority* brings the best return, i.e., they buy when noise traders depress prices and sell when noise traders push prices up. These "contrarian traders" are also called fundamentalists, sophisticated traders or α-investors [2–6,17,18]. They base their decisions not on market euphoria but on rational expectation, and they push prices toward fundamental values. Another classification of agents in terms of two basic financial market strategies has also been investigated using realistic models. These models considered the presence of two subgroups containing individuals who are optimistic or pessimistic about the future development of the market [2,3].

Using the two-state majority-vote model, we propose a model that simulates two interactions that drive the evolution of financial markets, (i) the "follow the majority" herd behavior of noise traders and (ii) the "follow the minority" fundamentalist behavior of the contrarian traders. In our model, the option (the spin variable) of an agent is influenced by the option of its neighbors and by the global magnetization. In our "global-vote" configuration, we combine sociophysics and econophysics to create an agent-based model with different investors to study the opinion dynamics in economic systems.

The remaining sections continue as follows. In Section 2 we describe the global-vote model for financial markets and introduce the relevant quantities used in our computational analysis. Section 3 summarizes the numerical results, and in Section 4 we present our conclusions.

2. The model

In our agent model we place the individuals into the nodes of a regular square lattice network of size $N = L \times L$. The option of one individual in a given time t is represented by a spin variable that can assume values of either $+1$ or -1. To simulate the composition of real-world financial markets, we define a fraction p to be contrarian individuals and a fraction $1-p$ to be noise traders, represented by spin variables α and λ, respectively. Contrarian traders are influenced by global magnetization, and noise traders are influenced by nearest neighbors. The two types of traders are randomly distributed in the interaction network and the option of a contrarian investor α_i flips with a probability

$$w(\alpha_i) = \frac{1}{2}\left[1-(1-2q)c_i\alpha_i \mathrm{sgn}\,(M)\right], \tag{1}$$

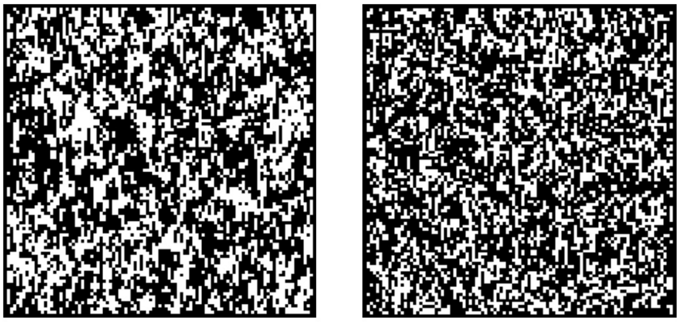
where $\mathrm{sgn}(x) = -1, 0, +1$ when $x < 0$, $x = 0$, and $x > 0$, respectively. The constant c_i is the strategy of the contrarian traders and $c_i = -1$ for all α individuals, indicating that contrarian traders tend to agree with the global minority traders. We denote $N_\alpha = Np$ to be the number of contrarian agents and $N_\lambda = N(1-p)$ to be the number of noise traders. Thus $N = N_\alpha + N_\lambda$ and we define the magnetization of the system to be

$$M = \frac{1}{N}\left(\sum_{i=1}^{N_\alpha}\alpha_i + \sum_{j=1}^{N_\lambda}\lambda_j\right), \tag{2}$$

The option of a noise trader λ_j flips with a probability

$$w(\lambda_j) = \frac{1}{2}\left[1-(1-2q)c_j\lambda_j \mathrm{sgn}\left(\sum_{\delta=1}^{k_j}\lambda_{j+\delta}\right)\right], \tag{3}$$

and the sum extends over the $k_j = 4$ neighbors of site j in a square lattice network. Here we set the strategy c_j to be equal to $+1$ for all λ investors, reflecting the trend that noise traders agree with the local majority. The variable q in Eqs. (1) and (3) is the noise parameter of the model, quantifies the temperature of the system, and adds randomness to agent decision-making process. When q increases, contrarian agents become the majority and noise traders the minority. For simplicity, we suppress the time variable t in α_i, λ_j and M.



(a) (b)

Fig. 1. Snapshots of a simulation on a square network of agents in the steady state. Here we have $q = 0.05$ with (a) $p = 0.2$ and (b) $p = 0.5$. In (a) and (b) we can observe clusters of ordered options and in (b) these structures are smaller. White dots denote spin up and black dots spin down.

3. Results and discussion

We perform Monte Carlo simulations on a lattice of linear size $L = 101$. Here one unity of time corresponds to N attempts to change the state of randomly selected network agents, measured in Monte Carlo Steps (MCS). We start the simulations with a random configuration of investor options, simulate 10^6 MCS and discard 100 MCS as the thermalization time.

When $p = 0$ we recover the original majority-vote model. When q is above zero, near $q_c \approx 0.075$ some clusters generate smaller interior clusters in a Matryoshka doll configuration and become a system with zero magnetization at q_c in the thermodynamic limit ($N \to \infty$) [9]. At higher values of q the system is fully disordered and magnetization approaches zero. When $p \neq 0$, system clusters emerge when the values of q drop below $q_c(p = 0)$ and the system also exhibit real-world market features. This configuration is the focus of our analysis.

Fig. 1 shows snapshots of $q = 0.05$ when the fraction of contrarian agents set at (a) $p = 0.2$ and (b) $p = 0.5$. Note the cluster formation occurring in (a), where the value of the noise parameter is below the critical value when $p = 0$ [9]. This indicates that the presence of contrarian agents causes the system to transition from order to disorder when q values are lower than $q_c(p = 0) \approx 0.075$. Further increasing the fraction of contrarian traders causes clusters to shrink or disappear and the system to become increasingly disordered, consistent with previous findings that noise traders increase stock market volatility and contrarian traders increase market stability [2,17,19].

Designating a buyer and a seller to have options $+1$ and -1, respectively, we relate the magnetization M to the aggregate excess demand for the asset at time t [4–6,20]. When the magnetization oscillates near zero, the number of buyers and sellers is approximately the same and the market fluctuates around equilibrium. We can also relate magnetization and global price changes because an aggregate excess demand impacts stock prices. If the excess demand is positive ($M > 0$) the prices rise, if negative ($M < 0$) they fall. Thus we use magnetization to quantify price and investigate the statistical properties of a financial time series in logarithmic returns and price changes. We define the logarithmic return at time t to be

$$r(t) = \log[|M(t)|] - \log[|M(t-1)|]. \tag{4}$$

Fig. 2 shows the systemic logarithmic returns with $q = 0.05$ for $p = 0.2$ and $p = 0.5$, which are the same values as those in Figs. 1(a) and 1(b), respectively. Fig. 2 shows ordered and turbulent phases of trader dynamics, reflecting the rapid agent option rearrangement. These return transitions are related to the metastable phases of spin model magnetization, as shown in Fig. 1(a). Fig. 2 also shows that when $p = 0.5$ there are oscillations in the logarithmic returns, but an order of magnitude smaller, related to the smaller cluster configuration showed in Fig. 1(b). We thus assign two model regimes when $q = 0.05$: a "strong market phase" for $p = 0.2$ and a "weak market phase" for $p = 0.5$. In Fig. 3 we show a comparison between the magnetization for these two phases. Note that the magnetization for the weak market is randomly distributed and becomes unstable for the strong market, denoting that the system is close to a metastable phase with high volatility. We find similar results for other values of the noise parameter q. Therefore, in this work, we investigate how the probability of contrarian traders p affects the market.

This logarithmic return behavior is the analogue to a bubble-related asset value that initially increases, seemingly without reason. A bubble is an economic cycle with a rapid increase in asset prices followed by a rapid decrease. It is caused by a surge in asset prices unwarranted by asset characteristics and is driven by emergent market behavior. When no more agents opt to buy it at the elevated price, a massive sell-off occurs and the bubble deflates. We see this behavior in the daily fluctuations in the Dow Jones Index and the Bitcoin exchange rate [5,21]. Fig. 4 shows a plot of the logarithmic return of Bitcoin prices from 16 June 2010 to 05 January 2018 for a qualitative comparison.

Volatility in finance is the degree of variation or dispersion around the average return of a trading price series over time. Fig. 2 shows that the high-volatility phases of the global-vote model are strongly clustered when $p = 0.2$. In this scenario investors become anxious, and strategic agents attempt to buy low and sell high. To characterize volatile market

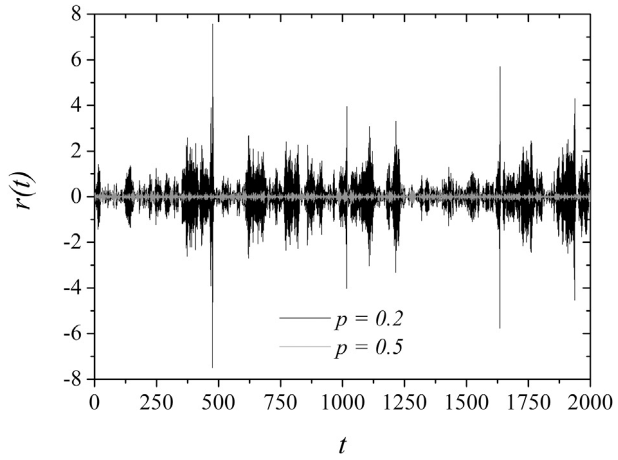


Fig. 2. Logarithmic relative change of the absolute value of the magnetization of the model for $q = 0.05$ with $p = 0.2$ (black) and $p = 0.5$ (gray). For this set of parameters, the market shows long periods of inactivity with bursts of activity and the values in gray are an order of magnitude smaller.

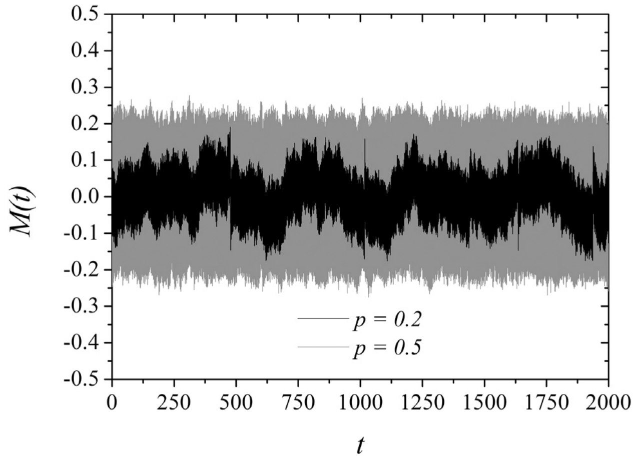


Fig. 3. Magnetization of the model versus time for $q = 0.05$ with $p = 0.2$ (black) and $p = 0.5$ (gray).

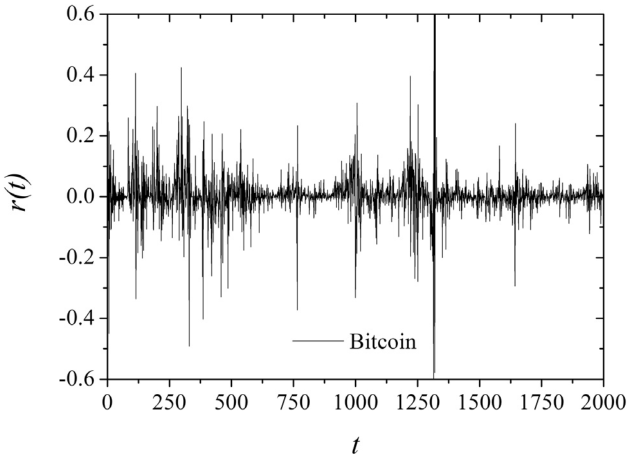


Fig. 4. Logarithmic returns of the exchange rate of Bitcoin from 16 June 2010 to 05 January 2018.

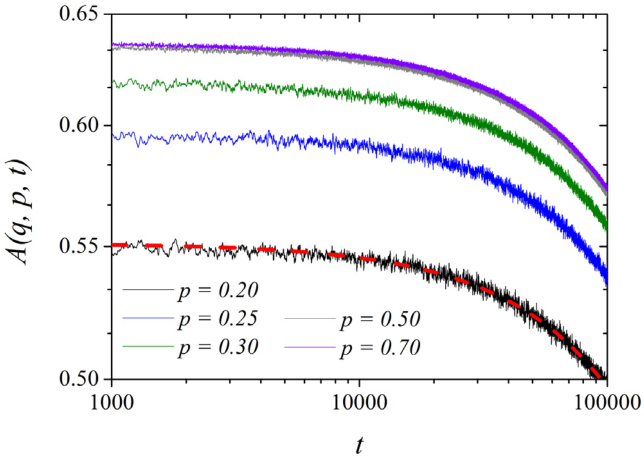


Fig. 5. Log–log plot of the autocorrelation function of the absolute log-returns versus time t for $q = 0.05$. From bottom to top we have $p = 0.20, 0.25, 0.30, 0.50$ and 0.70. The dashed red line is the exponential fit $A(q, p, t) \sim exp[-(1.0 \times 10^{-6})t]$ for the data.

dynamics and quantify volatility clustering, we calculate the corresponding autocorrelation of absolute returns. We define this autocorrelation

$$A(\tau) = \frac{\sum_{t=\tau+1}^{T} [|r(t)| - |\bar{r}|][|r(t-\tau)| - |\bar{r}|]}{\sum_{t=1}^{T} [|r(t)| - |\bar{r}|]^2}, \tag{5}$$

where τ is the time-step, $r(t)$ the current value of the return, $\bar{r}$ the average return value and T the total time simulated. As observed in real financial markets, the global-vote model shows that the autocorrelation of returns disappears quickly as time increases, implying that the returns themselves are uncorrelated [22]. However the autocorrelation function of the absolute returns, defined by Eq. (5), display a positive long-time correlation with a slow decay. This behavior refers to the observation by Mandelbrot [23] that "large changes tend to be followed by large changes of either sign, and small changes tend to be followed by small changes".

We see this slow decay of the autocorrelation of absolute log-returns when the data display volatility clustering. Fig. 5 shows the autocorrelation of the absolute returns for $q = 0.05$ and several values of p, where the total time $T = 10^6$ MCS and $1 \leq \tau \leq 10^5$ MCS. Fig. 2 shows that when $p = 0.2$ (black) the system is in a strong market phase. The rapid variations in the returns are reflected in an exponential decay in the autocorrelation of absolute log-returns, in agreement with the Bornholdt model that also shows a similar decay for this function [4–6]. To quantify this effect we fit the autocorrelation function of absolute log-returns by the relation $A(q, p, t) \sim exp(-t/t_0)$, where we find that $1/t_0 \approx 1.0 \times 10^{-6}$ (dashed red line). As p increases, the autocorrelation curves continue to exponentially decay with time, but the oscillation of logarithmic returns is significantly smaller than those for $p = 0.2$, as seen in the gray line in Fig. 2. Nevertheless, the other values of p in Fig. 5 can be fit using decreasing exponentials where all the exponents are close to 1.0×10^{-6}.

Fig. 6 plots the graphical shape of the probability distribution of the logarithmic returns as its frequency for $q = 0.05$ and several values of the fraction p of contrarian agents. As in real-world market, the distributions have fat tails. We perform a statistical analysis of our data and find that the skewness $S(p)$, which is zero for normal distributions, it is equal to $S(0.2) = -0.0135$ and $S(0.5) = 0.0016$ in strong and weak market phases, respectively. Here positive skewness indicates investment opportunities because the right tail of the distribution is fatter than the left, and positive returns tend to occur more often. We also calculate the kurtosis $\kappa(p)$ of the data and for a random sample from a normal distribution $\kappa = 3$. We find that when $q = 0.05$, $\kappa(0.5) = 3.04$, and we conclude that the logarithmic returns of the weak market are distributed as a Gaussian distribution. However, when $q = 0.05$ and $p = 0.2$, we find $\kappa(0.2) = 6.33$ for the strong market phase. This positive excess kurtosis indicates that the distribution of the logarithmic returns is leptokurtic and has fat tails which are consistent with the well-known empirical finding that the distribution of logarithmic returns in real-world financial markets is non-Gaussian [24,25].

To analyze the fat tails of our distributions graphically, we use a comparative quantile–quantile (QQ) plot [26]. Fig. 7 shows QQ-plots for the distribution of logarithmic returns in strong and weak markets, ranging from $p = 0.20$ to $p = 0.70$ when $q = 0.05$, and we test all of them against the normal distribution. The red line is the reference line $y = x$, and when two distributions are similar, the points in the QQ plot lie approximately on $y = x$. When the market is strong, the cross shape is no longer linear, indicating that the tails of the distribution are fat and the distribution of logarithmic returns for the global-vote model with $q = 0.05$ and $p = 0.2$ is non-Gaussian. We see a similar behavior in S&P 500 index of daily returns [24]. In contrast to a normal distribution, in a fat tailed distribution there is a higher probability that the values of the logarithmic returns are extreme. There are several proposed shapes for the distribution of (logarithmic) returns in financial markets, including the generalized hyperbolic Student-t, the normal inverse Gaussian, the exponentially truncated stable,

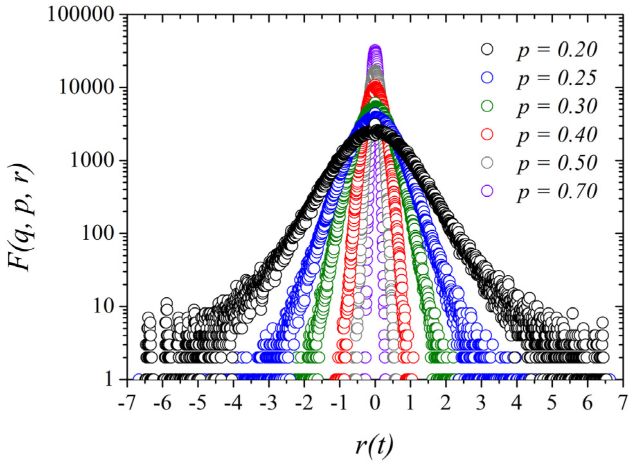


Fig. 6. Heavy-tailed distributions of log-returns for $q = 0.05$ with $p = 0.20, 0.25, 0.30, 0.40, 0.50$ and 0.70 in 10^6 MCS.

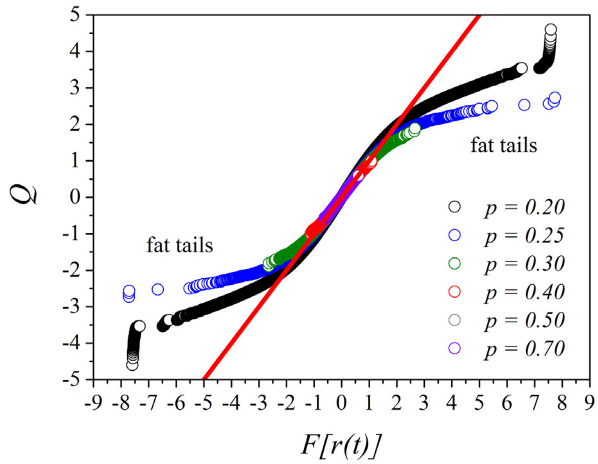


Fig. 7. Quantile–quantile plot for the strong market phase, where $q = 0.05$ and $p = 0.20, 0.25, 0.30, 0.40, 0.50$ and 0.70. Horizontal axis represents the normal theoretical quantile Q and vertical axis represents the data quantile or frequency of returns $F[r(t)]$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and the asymmetric Laplace distribution, but no consensus has been reached on the exact form of the tails in real-world financial markets [25,27].

As the fraction of fundamentalist traders increases the log-return distribution transitions from a heavy-tail decay to a exponential decay. We find that the log-returns for $p < 0.5$ are well fit by the Student's t distribution and Fig. 8 shows a log–log plot of the absolute log-returns and the Student's t fit to the data. For $p = 0.50$ and 0.70 we use a Gaussian fit, once the rate of decay is specified by the degree of freedom ν and generalizes the exponential decay of Gaussian distribution ($\nu \to \infty$) with heavy-tail distributions ($0 < \nu < \infty$). Assuming the mean μ is zero (see Fig. 9), the Student's t distribution $f(r; \nu, \mu, \sigma) = f(r; \nu, \sigma)$ with degree ν and scale σ for the log-returns values r is

$$f(r; \nu, \sigma) = \frac{1}{\sqrt{\nu\sigma^2} B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \left(1 + \frac{r^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}, \tag{6}$$

where

$$B\left(\frac{\nu}{2}, \frac{1}{2}\right) = \int_0^1 t^{\nu/2-1}(1-t)^{-1/2}\mathrm{d}t, \tag{7}$$

is the Beta function. Fitting the log-returns with Student's t distribution for $p = 0.20, 0.25, 0.30$ and 0.40 we obtain the degree-scale pair (ν, σ) for $p = 0.20, 0.25, 0.30$ and 0.40 shown in Table 1. As the degree of freedom ν increases with p, we

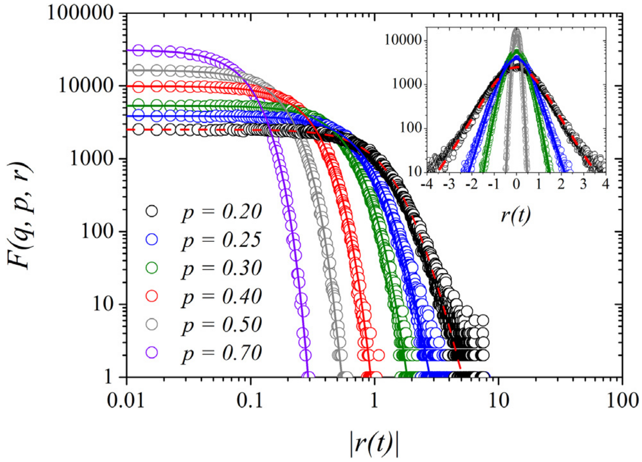


Fig. 8. Frequency $F(q, p, r)$ versus the absolute value of logarithmic returns $|r(t)|$ for $q = 0.05$ with $p = 0.20, 0.25, 0.30, 0.40, 0.50$ and 0.70 in 10^6 MCS. The dashed red line is a Student't fit with degree $\nu = 6.6 \pm 0.1$ and scale $\sigma = 0.7624 \pm 0.0008$ to the strong market regime of $p = 0.20$.

Table 1
Dependence of the degree ν and scale σ on the fraction p for $q = 0.05$.

Fraction p	0.20	0.25	0.30	0.40
Degree ν	6.6 ± 0.1	12.3 ± 0.3	21.2 ± 0.8	53 ± 4
Scale σ	0.7624 ± 0.0008	0.5075 ± 0.0004	0.3689 ± 0.0003	0.2013 ± 0.0001

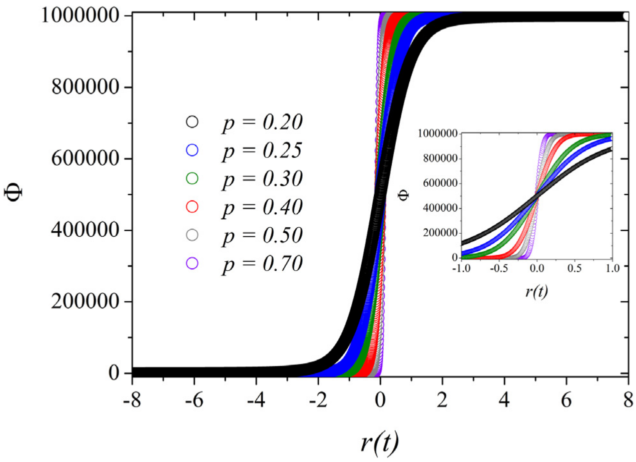


Fig. 9. Cumulative distribution function Φ for the logarithmic returns $r(t)$ calculated in 10^6 MCS for several values of the contrarians concentration p. In the inset we show the details of the distributions around zero.

find that the distribution of the absolute log-returns for $p = 0.50$ and 0.70 are fitted using a Gaussian function $g(r; \sigma)$ with zero mean and variance $\sigma(p) = \sigma$

$$g(r; \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(r/\sqrt{2}\sigma)^2}, \tag{8}$$

where $\sigma(0.50) = 0.12169 \pm 0.00007$ and $\sigma(0.70) = 0.06393 \pm 0.00004$. In Fig. 9 we show the cumulative distribution Φ for the log-returns where we observe that the mean is zero for all values of p investigated. The relatively small values for the variance σ yield a step function shape for the cumulative Gaussian function when $p = 0.50$ and 0.70.

The distribution of the absolute logarithmic stock returns $|r(t)|$ exhibits power-law asymptotic behavior

$$F(|r(t)|) \sim |r(t)|^{-\theta}, \tag{9}$$

where θ is between 2 and 4 for real-world financial markets [1,28,29]. The Student's t is a multifractal distribution with an asymptotic tail decay and the negative slope of the log–log plot is computed to be

$$\frac{\mathrm{d}}{\mathrm{d}[\log(r)]} f(r; \nu, \sigma) = \frac{r^2(\nu+1)}{r^2 + \nu\sigma^2}, \tag{10}$$

and in the limit $r^2 \gg \sigma^2$, we obtain $f(r|\nu, \sigma^2) \sim r^{-(\nu+1)}$. Thus numerical fits estimating just the tail of the distribution tend to underestimate the asymptotic slope [30,31]. For the fraction of fundamentalist traders p between 0.2 and 0.4 we observe power-law behavior and we estimate the asymptotic slope given by $\theta(p) = \nu(p) + 1$ to be $\theta(0.20) \approx 7.6$, $\theta(0.25) \approx 13$, $\theta(0.30) \approx 22$ and $\theta(0.40) \approx 54$. For the degree of freedom ν around and above 50, the distribution closely approximates the Gaussian distribution.

In Fig. 8 we also note that the point on the horizontal axis corresponding to the scale parameter σ value indicates the region where the slope begins to decrease significantly below zero. For values of the absolute logarithmic return above the scale value the slope is continuously changing until it saturates in the asymptote. In particular, for a truly scale-free distribution the scale parameter σ converges to zero and there is a minimal curvature in the slope.

We conclude that the global-vote model can qualitatively reproduce real-world financial market behavior when the noise parameter q and the contrarian agents concentration p are properly tuned. We can also use its results to estimate the distribution of agents and characterize the mechanisms that drive the behavior of interacting agents in economic systems.

4. Conclusion and final remarks

We have proposed an agent-based model of a stock market with heterogeneous traders. Our model has two kinds of investor – contrarian agents and the noise traders – who are influenced differently by market interactions when they update their market options. Contrarian agents are influenced by global market magnetization, and noise traders are influenced by local market magnetization, i.e., the behavior of nearest neighbors. We demonstrate that trading volume and market price are related to the magnetization in our model. Because of this relation, the competition between local and global interactions in our global-vote model reproduces such qualitative stylized facts of real-world financial data as clustered volatility, long-time correlated slow decay, and non-Gaussian and power-law distributed returns. We have shown that the distribution of logarithmic returns is well modeled by the Student's t distribution and that the degree of freedom increases as the percentage of contrarian traders increases. Despite its simplicity, it combines two fields of the statistical mechanics – econophysics and sociophysics – and allows us to better understand how trading behavior effects the volatility of financial markets.

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