Renormalization-group calculation of the critical-point exponent $\eta$ for a critical point of arbitrary order

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The critical-point exponent $\eta$ for a critical point of order $\mathcal{O}$ in dimensions less than $d_c \equiv 2\mathcal{O}/(\mathcal{O} - 1)$, is calculated to leading nonvanishing order in the parameter $\epsilon^2 \equiv d_0 - d$. The result is given for $n$-component isotropically interacting magnetic systems. For Ising systems, $n = 1$, the result is $\eta_0 = \epsilon^2(\mathcal{O} - 1)^2/(2\mathcal{O})^3$. As $\mathcal{O}$ increases, the coefficient of $\epsilon^2$ rapidly becomes very small, varying as $2^{-\mathcal{O}}$ for large. In the limit of large $n, \eta_0$ for odd order points approaches a constant and, for even order points, is proportional to $1/n$.

The classification and study of critical points of “higher order” has been of recent interest. The order of a critical point is defined by some authors to be the number of phases simultaneously critical at the critical point. Thus, an ordinary critical point is an $\mathcal{O} = 2$ point; tricritical points are $\mathcal{O} = 3$ points. Although there are many kinds of higher-order points, much of the work has concentrated on systems that in the mean-field approximation could be represented by a Landau-Ginzberg form for the Hamiltonian density,

$$ H(\mathbf{s}) = -\int d^d x \left( \frac{1}{2} |\nabla \phi(x)|^2 + \sum_{\mathbf{k} \neq 0} \frac{\mu_k}{(2\mathcal{O}!)^2} \langle \mathbf{s} \cdot \mathbf{s} \rangle \right), \tag{1} $$

where we have specialized to the “magnetic” case of an isotropically interacting $n$-component spin $\mathbf{s}(\mathbf{x})$.

The renormalization-group approach to such systems was introduced by Wilson at $\mathcal{O} = 2$. Corrections to mean-field behavior are calculated in a perturbation expansion in $\epsilon^2 \equiv 4 - d$. The tricritical case, $\mathcal{O} = 3$, has been studied by Riedel and Wegner at $d = 3$. Chang, Tuthill, and Stanley and Stephen and McCauley calculated exponents below three dimensions in an expansion in $\epsilon^2 \equiv 3 - d$. Reference 5 also gave explicit exponents to first order in $\epsilon^2 \equiv 3 - d$ for the $\mathcal{O} = 4$ case. The critical-point exponents for the general $\mathcal{O}$ case were given in Nicoll, Chang, and Stanley at first order in $\epsilon^2 = 2\mathcal{O}/(\mathcal{O} - 1) - d$. The critical-point exponent $\eta$ was shown in Ref. 7 to be at most $O(\epsilon^2)$. In this work, we complete the calculation of all critical-point exponents to leading order by calculating $\eta$ to $O(\epsilon^2)$.

The $\epsilon^2$ calculations of Ref. 7 were based on the differential renormalization-group generator of Wegner and Houghton. The calculation of $\eta$ by this method is difficult and, therefore, through most of this work we will adopt a field-theoretic approach utilizing Feynman diagrams. However, we will extract the dependence of $\eta$ on the number of spin components $n$ by combining graph counting with the solutions of Ref. 7.

Following the method used to locate fixed points, we assume $\mu_k$ to be $O(\epsilon^2)$ for $k = \mathcal{O}$. It is then possible to carry out a self-consistent perturbation expansion in the parameters $\mu_4, \mu_6, \ldots, \mu_{2\mathcal{O}}$ while applying a mass counterterm that so that the bare propagator is $(p^2 + \mu)^{\mathcal{O}}$, with $r^{-1}$ the zero-order ing-field susceptibility. The exponent $\eta_0$ is defined by a proportionality relation for the Fourier transform $G$ of the spin-spin correlation function for small wave number,

$$ G^{-1}(p) \sim p^{2-m} \sim p^2 (1 - \eta_0 \log \cdots), \tag{2} $$

at the order-$\mathcal{O}$ point ($r = 0$). We will now show that to $O(\epsilon^2)$, the calculation of $\eta_0$ involves only two Feynman graphs to be evaluated in dimension $d_c$.

In the perturbation expansion for $G^{-1}$ we may write

$$ G^{-1}(p) \sim 1 + \Sigma(p, r), \tag{3} $$

where $\Sigma$ represents the sum of all remaining graphs (with counterterm) displayed schematically in Fig. 1. The mass counterterm $\mu_k - r$ cancels all $p$-independent terms in (3) (in particular, all single-vertex diagrams). The series may be further simplified by formally eliminating closed loops that include only one vertex and introducing $r$-dependent generalized vertices $\Gamma_{2k} (r)$, defined by

$$ \Gamma_{2k}(r) = \mu_k + \sum_{L \neq 0} \frac{1}{12} [F(r)]^L. \tag{4} $$

Here, as in Ref. 6, $F(r)$ represents the loop integral $\int d^d p G(p, r)/(2\pi)^d$. With this change in notation and to $O(\epsilon^2)$, the set of graphs in $\Sigma$ is reduced to those shown in Fig. 2.

Next, we note that $\Gamma_{2k}(r = 0) = 0$ for all $k \neq 0$. This follows from Wilson's scaling theorem for $2k$-point vertex functions

$$ \Gamma_{2k}(p = 0) \sim r^{2-(k-1)d/(2-m)}. \tag{5} $$

For $d \leq d_c$, Eq. (5) requires that $\Gamma_{2k}$ vanish at $r = 0$ for all $k < 0$. Since the first-order perturba-
The perturbation series to $O(\epsilon^2)$ for the function $\Sigma(\bar{p}, r)$, defined by Eq. (3). Each diagram carries net momentum $\bar{p}$.

\[ \Sigma(\bar{p}, r) = \frac{\epsilon^2}{u_{2k}} + \ldots \]

**FIG. 1** Perturbation series to $O(\epsilon^2)$ for the function $\Sigma(\bar{p}, r)$, defined by Eq. (3). Each diagram carries net momentum $\bar{p}$.

The combinatorial factor for this diagram may be evaluated by considering first the Ising case, in which it is simply $1/(2\Theta - 1)!$. To determine the $n$ dependence, it suffices to note that a factor of $n + 2N - 2$ is associated with the connection of two legs of a single $2N$-leg vertex. Thus, the $n$ dependence of the $u_{2k}$ diagram is given by $f_1(n)/(2\Theta - 1)!$, where

\[ f_1(n) = \prod_{k=1}^{n-1}(n+2k)/(2k+1). \]

With this factor and denoting the $u_{2k}$ integral by $I_k$, the correspondence between (2) and (3) gives

\[ p^2(1 - \eta_\Theta \ln p) = \epsilon^2 \frac{u_{2k} f_1(n)}{(2\Theta - 1)!} \times [I_k(p, r = 0)](\Theta - 1)/2) [\epsilon_\Theta \ln r \cdots]. \]

Since $u_{2k}$ is $O(\epsilon^2)$, $\eta_\Theta$ is clearly $O(\epsilon^2)$.

The fixed point value of $u_{2k}$ remains to be determined; it is chosen so that the vertex functions satisfy scaling laws. For $k \neq 0$ in (5) this gives

\[ \Gamma_{2k} \sim \frac{n(n-1)/2 - 1 + \epsilon_\Theta [(\Theta - 1)/2] \ln r \cdots. \]

The constant of proportionality must also be expanded as a series in $\epsilon_\Theta$, so that

\[ \Gamma_{2k} = A + \epsilon_\Theta [A(\Theta - 1)/2] \ln r + B, \]

with $A$ and $B$ constants.

In first order, $\Gamma_{2k}$ is $u_{2k}$, so that $A = u_{2k}$. Second-order terms all involve two-vertex diagrams $u_{2k}$, $u_{2k'}$ (with $l, l' \leq \Theta$) and graphs with internal lines numbering $l \leq \Theta$ (cf. Fig. 3). The $r$ dependence of $u_{2k}$ is given by the integral $F(r) - F(0)$, since by the remarks above $u_{2k}(0) = 0$ for $l < \Theta$:

\[ F(r) - F(0) = \int \frac{d^d k}{(2\pi)^d} \left( \frac{1}{k^2 + r} - \frac{1}{k^2} \right) \]

\[ = -\frac{\Omega_d}{(2\pi)^d} \int_0^\infty \frac{k^{d-3}}{k^2 + r} dk, \]

where $\Omega_d = 2(\pi)^{d/2}/\Gamma(\frac{d}{2})$ is the area of the unit sphere in dimension $d$. Changing variables, we have

\[ F(r) - F(0) = -\frac{\Omega_d}{(2\pi)^d} \int_0^\infty \frac{dx x^{d-3}}{1 + x^2} \]

The integral converges for $2 < d < 4$ so that all $r$ dependence is in the prefactor; no $\ln r$ factors are present.

Next, we examine the $r$ dependence of the graph of Fig. 3. By power counting, this integral diverges like $r^{(d-3)/2}$ for small $r$. For $i < \Theta$, the integral converges at large $k$ without a momentum cutoff, and a change of variables similar to that in (11) shows that the diagram gives a prefactor of $r^{(d-3)/2}$ multiplied by a convergent integral. Only for $i = \Theta$ will $\ln r$ terms arise; the integral for this case is denoted as $I_2(r)$.

To compare with the scaling form (9), we note that the perturbation expansion gives

\[ \Gamma_{2k} = u_{2k} - \frac{(2\Theta) I_2(r)}{(2\Theta - 1)!} u_{2k} \cdots. \]

The resulting value for $u_{2k}$ to first order is

\[ u_{2k} = -\frac{(\Theta - 1)(\Theta)^2 \epsilon_\Theta}{(2\Theta - 1)!} \frac{I_2(r)}{I_2(r)_{\text{large}}} + \cdots. \]

Combining (13) with (7), the expression for the exponent $\eta_\Theta$ for $n = 1$; $n$ dependence will be discussed below) to leading order is

\[ \eta_\Theta = \epsilon_\Theta \frac{(\Theta - 1)^2 (\Theta)^2 I_2(r)_{\text{large}}}{(2\Theta - 1)! \Gamma(\theta) I_2(r)_{\text{large}}}. \]

All that remains is the calculation of the two integrals.
\[ I_1 = \int d^d x e^{i \Phi_R} \left( \frac{d^d x}{(2\pi)^d} \right)^{2\nu - 1} \]  
\[ I_2 = \int d^d x e^{i \Phi_R} \left( \frac{d^d x}{(2\pi)^d} \right)^{\nu} \]  
where \( d \) and \( \Phi \) are, of course, related by \( d = d_0 = 20/(e - 1) \). Both integrals are divergent as written; \( I_1 \) diverges quadratically and \( I_2 \) diverges logarithmically. To extract the finite terms desired, we cut off the \( R \) integrations by integrating over \( |R| > \Lambda^{-1} \).

From Bateman\(^{10}\) we note that
\[ \int d^d x e^{i \Phi_R} = \int_0^\infty dx x^{d-1} \frac{\Omega d}{(2\pi)^d} J_d(xq) \left( \frac{1}{x} \right)^{\nu}, \]  
where \( \nu = \frac{1}{2} (d - 2) \). Therefore, applying (17) to (15) we have
\[ I_1 = \left( \frac{\Omega d}{(2\pi)^d} \right)^{\nu} \int_0^\infty dx x^{d-1} J_d(Rp) \left( \frac{1}{x} \right)^{\nu} \]  
\[ \times \left[ \int_0^{\Lambda^{-1}} dk J_d(kR) \right]^{2\nu - 1}. \]  
(18)

The inner integral can be evaluated exactly; after a change of variable (18) becomes
\[ I_1 = \left( \frac{\Omega d}{(2\pi)^d} \right)^{2\nu - 1} \frac{\Omega d}{(2\pi)^d} R^2 \]  
\[ \times \int_0^\infty dx x^{d-1} J_d(x) \left( \frac{1}{x} \right)^{\nu}. \]  
(19)

The integral over the interval \([1, \infty)\) gives a finite contribution to the \( p^2 \) term. The integral over the interval \([\rho/\Lambda, 1]\) can be evaluated by expanding the Bessel function in its Taylor series. We find that
\[ I_1 = \left( \frac{\Omega d}{(2\pi)^d} \right)^{2\nu - 1} \frac{\Omega d}{(2\pi)^d} R^2 \]  
\[ \times \int_0^\infty dx x^{d-1} J_d(x) \left( \frac{1}{x} \right)^{\nu} \]  
\[ \times \frac{\Omega d}{4 \Gamma(\frac{1}{2} d)} \]  
\[ \times \frac{\rho^2}{\Gamma(\frac{1}{2} (d + 1))} \ln^2 R^2 + O(\Lambda^2). \]  
(20)

With these combinatorial factors, the result for general \( n \) and general \( \theta \) is
\[ \eta_\theta = \frac{4}{\Omega d} \left( \frac{1}{(2\pi)^d} \right)^{\nu} f_\theta(n) \]  
\[ \times \frac{\rho^2}{\Gamma(\frac{1}{2} d)} \frac{\rho^2}{\Gamma(\frac{1}{2} (d + 1))} \ln^2 R^2 + O(\Lambda^2). \]  
(26)

It is easy to check that (26) reduces to the previously calculated results for \( \theta = 2 \) and \( \theta = 3 \),
\[ \eta_\theta = \frac{\rho^2}{2(n + 2)^2}, \]  
\[ \eta_\theta = \frac{\rho^2}{2(n + 3)^2}. \]  
(27)

We note that as \( \theta \) increases the coefficient of \( \rho^2 \) rapidly becomes very small, \( \approx 2^{-6\theta} \) for \( \theta \) large. In the limit of large \( n \), \( \eta_\theta \) for odd order points approaches a constant and, for even order points, is proportional to \( 1/n \).

For all \( \theta > 3 \) we have \( d_\theta \approx 3 \), and the mean-field result, \( \eta_\theta = 0 \), therefore applies in three-dimension-
al systems. However, these results and those of
Refs. 5–7 may apply to higher-order critical points
in two-dimensional systems. In any event, the
previously obtained results for ordinary critical
points are placed in a broader theoretical context
by the extension to general \( \varnothing \).

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trary \( \varnothing \). Here \( \sigma \) is defined through the interaction
\( 1/\sigma^{\delta_\sigma} \) and Ref. 7 corresponds to calculations for \( \sigma \approx 2 \).