ELSEVIER

Physica A 233 (1996) 31–38

PHYSICA A

Ising model on the Sierpiński gasket: thermodynamic limit versus infinitesimal field

Tatijana Stošić[a,*], Borko D. Stošić[b,a], Sava Milošević[c], H. Eugene Stanley[d]

[a] *Laboratory for Theoretical Physics, Institute for Nuclear Sciences, Vinča, P.O. Box 522, YU-11001 Belgrade, Yugoslavia*

[b] *Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife-PE, Brazil*

[c] *Faculty of Physics, University of Belgrade, P.O. Box 550, YU-11001 Belgrade, Yugoslavia*

[d] *Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA*

Received 24 May 1996

Abstract

Owing to extremely slow decay of correlations, the limit $H \to 0$ presents a poor approximation for the Ising model on the Sierpiński gasket. We present evidence of the competitive interplay between finite size scaling and thermodynamic scaling for this model, where both finite size and finite field induce an apparent phase transition. These observations may be relevant for the behavior of porous magnetic materials in real laboratory conditions.

The Ising model on the Sierpiński gasket presents one of the rare exactly solved [1,2] models in statistical physics in the case of nonzero external magnetic field. The critical behavior was exactly deduced [1] in the thermodynamic limit, for $H \to 0^-$, where it was shown that the model exhibits a phase transition only at zero temperature. Nevertheless, this model has an unusual property that the nature of magnetic phase transitions deduced for an infinite system is not applicable to systems of laboratory dimensions [3]. For the same reason why it is relevant to ask [4] how finite size affects thermodynamic behavior of real samples, the question should be posed how the presence of small fields (unavoidable in real laboratory conditions) may affect measurements. Another unusual property of the Ising model on the Sierpiński gasket that up to date went unnoticed, is that *extremely* low field causes magnetic ordering in a *wide* temperature range. In this paper we discuss the fine competition between finite size scaling and scaling with the field, which should both be quite relevant in laboratory

* Corresponding author.

PII S0378-4371(96)00240-3

conditions. In real experiments involving porous magnetic materials or other materials with (stochastic) fractal structure, similar considerations may also be necessary.

We consider a Sierpiński gasket (SG) with an Ising spin at each vertex, described by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle nn \rangle} S_i S_j - H \sum_i S_i \,, \tag{1}$$

where J is the coupling constant, $S_i = \pm 1$ is the spin at site i, H is the external magnetic field, and $\langle nn \rangle$ denotes summation over the nearest neighbor pairs. SG itself is constructed in stages, the zeroth stage being a simple equilateral triangle of unit side length. In each construction stage n, three triangular structures from the previous stage are joined by their vertices into a larger triangular structure. At each stage the partition function is given by

$$Z = Z_1 + 3Z_2 + 3Z_3 + Z_4 \,, \tag{2}$$

where Z_1, Z_2, Z_3, and Z_4 are partial partition functions corresponding to $\{+++\}$, $\{+-+\}$, $\{-++\}$, and $\{---\}$ configurations of the vertex spins, respectively. For the zeroth stage of construction we have

$$Z_1 = e^{3\beta J + 3\beta H} \,, \tag{3a}$$

$$Z_2 = e^{-\beta J + \beta H} \,, \tag{3b}$$

$$Z_3 = e^{-\beta J - \beta H} \,, \tag{3c}$$

$$Z_4 = e^{3\beta J - 3\beta H} \,, \tag{3d}$$

where $\beta = 1/k_B T$, and k_B is the Boltzmann constant. The exact recursive relations between the partial partition functions at two consecutive stages of construction are given by [5]

$$Z_1' = Z_1^3 e^{-3\beta H} + 3Z_1 Z_2^2 e^{-\beta H} + 3Z_2^2 Z_3 e^{\beta H} + Z_3^3 e^{3\beta H} \,, \tag{4a}$$

$$\begin{aligned} Z_2' = {} & Z_1^2 Z_2 e^{-3\beta H} + Z_2^3 e^{-\beta H} + 2Z_1 Z_2 Z_3 e^{-\beta H} \\ & + 2Z_2 Z_3^2 e^{\beta H} + Z_2^2 Z_4 e^{\beta H} + Z_3^2 Z_4 e^{3\beta H} \,, \end{aligned} \tag{4b}$$

$$\begin{aligned} Z_3' = {} & Z_1 Z_2^2 e^{-3\beta H} + Z_1 Z_3^2 e^{-\beta H} + 2Z_2^2 Z_3 e^{-\beta H} \\ & + 2Z_2 Z_3 Z_4 e^{\beta H} + Z_3^3 e^{\beta H} + Z_3 Z_4^2 e^{3\beta H} \,, \end{aligned} \tag{4c}$$

$$Z_4' = Z_2^3 e^{-3\beta H} + 3Z_2 Z_3^2 e^{-\beta H} + 3Z_2^2 Z_4 e^{\beta H} + Z_4^3 e^{3\beta H} \,. \tag{4d}$$

Since the aim of this paper is to discuss the competition between the thermodynamic limit and the limit $H \to 0^+$ on the basis of extremely high precision numeric calculations, it is important to emphasize here that recursive relations corresponding to (4)

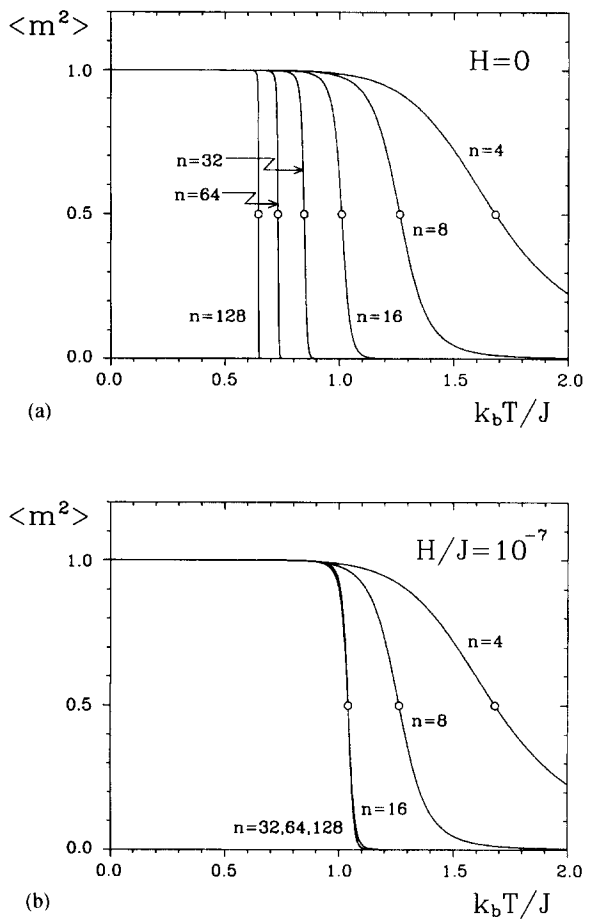

Fig. 1. Temperature dependence of the order parameter $\langle m^2 \rangle$ for different steps of construction $n = 4, 8, 16, 32, 64, 128$ of the Sierpiński gasket for (a) zero field and (b) in the field $H/J = 10^{-7}$ (corresponding to $H/g\mu_B \sim 0.22$ G for coupling constant $J/k_B = 300$ K). Finite size induces an apparent phase transition, vanishing only in the true thermodynamic limit (see Ref. [3]). It is seen that the presence of a small field inhibits the shift towards zero temperature, with increasing system size, of the ordered phase region.

given in Refs. [2,3] are *not exact* for finite systems in nonzero field. In both references, the field dependent multiplicative terms in (4) (corresponding to the fact that the energy at a given stage of construction is not a simple sum of energies of the three constituent previous stage structures) have been neglected, which is strictly valid only after one of the two limits has been reached. Differentiating (4) with respect to field, one obtains [5] recursive relations for the field derivatives of the partition function, which can be numerically iterated for arbitrary set of parameters J, H, and T.

From the numerical viewpoint, iteration of recursive relations is not demanding in either computer speed nor memory. However, it turns out that as n increases, the result of numerical iterations becomes increasingly dependent on the precision used.

Highest available precision of Fortran, REAL*16 (128 bits distributed between mantissa and exponent provide a maximum of 38 decimal digits for the mantissa) becomes insufficient for large sizes and small values of the field. To overcome this difficulty, we have used "Mathematica" symbolic language which has arbitrary available precision. In Fig. 1 we present numerical results for the temperature dependence of $\langle m^2 \rangle$, for $n = 4, 8, 16, 32, 64, 128$ (precision of roughly 100 decimal digits proved to be sufficient for all our calculations). Fig. 1(a) ($H = 0$) corresponds to the results of Liu [3], where it was shown that $n \to \infty$ is a poor approximation for laboratory size systems. Apparent magnetic transition temperature $T_m(L, H)$, defined as the point where $\langle m^2 \rangle_L = 1/2$, was found [3] to have limiting behavior

$$T_m(L \to \infty, H = 0) \sim \frac{1}{\ln \ln L}, \tag{5}$$

where $L = 2^n$ is the linear size of the gasket. In Fig. 1(b) we present corresponding results for a very small field $H/J = 10^{-7}$. It follows that the (weak) dependence of T_m on L is suppressed by the presence of the small field beyond $n = 16$, and that even in the thermodynamic limit the system should remain ordered up to $T_m \sim J/k_B$!

In Fig. 2 we present analogous results for different values of field $H/J = 10^{-2}$, 10^{-4}, 10^{-8}, 10^{-16}, 10^{-32}, 10^{-64}. Fig. 2(a) compared with Fig. 1(a) stresses the analogy between "finite size" and "finite field" (infinite system size here corresponds to iterating recursive relations beyond the point where the curves stop changing with further iterations). It follows that $H \to 0^+$ also presents a poor approximation for this model, that is, the system remains ordered up to temperatures of the order of $k_B T/J \sim 1$ in the presence of inconceivably small fields. In analogy with Fig. 1(b), in Fig. 2(b) we present temperature dependence of $\langle m^2 \rangle$ for $n = 16$, and different field values. It follows that the (weak) dependence of T_m on H is suppressed by finite size beyond $H/J = 10^{-8}$.

Following Liu [3] let us now assume that we are dealing with a real sample and assign $a = 3$ Å to the lattice constant and $J/k_B = 300$ K (this value of the coupling constant corresponds to a material with transition temperature of roughly 1000 K on the triangular lattice). Further taking $g = 2$ for the Lande factor, and using values $\mu_B = 0.927 \times 10^{-20}$ erg/G and $k_B = 1.381 \times 10^{-16}$ erg/K for the Bohr magneton and the Boltzmann constant, respectively, we have $H/g\mu_B \sim 0.22$ G for the field of Fig. 1(b), which is smaller than the field of the earth ($\sim$1 G). We can conclude that a laboratory sample with SG structure larger than $L = 2^{16}a \sim 6.5 \times 10^{-4}$ cm should exhibit an apparent phase transition at roughly room temperature, in the field of the earth. If the field of the earth is compensated and experiment performed at the order of 1 mG, the apparent transition temperature should become size independent for sample sizes exceeding $L = 2^{20}a \sim 0.1$ mm.

To gain a better understanding of the competition between the thermodynamic limit and the limit of zero field, let us now turn to the scaling arguments of Ref. [1] which predict that magnetization should scale as

$$m(L, T, h) = \mathcal{M}\left(L/\xi, \xi^D h\right), \tag{6}$$

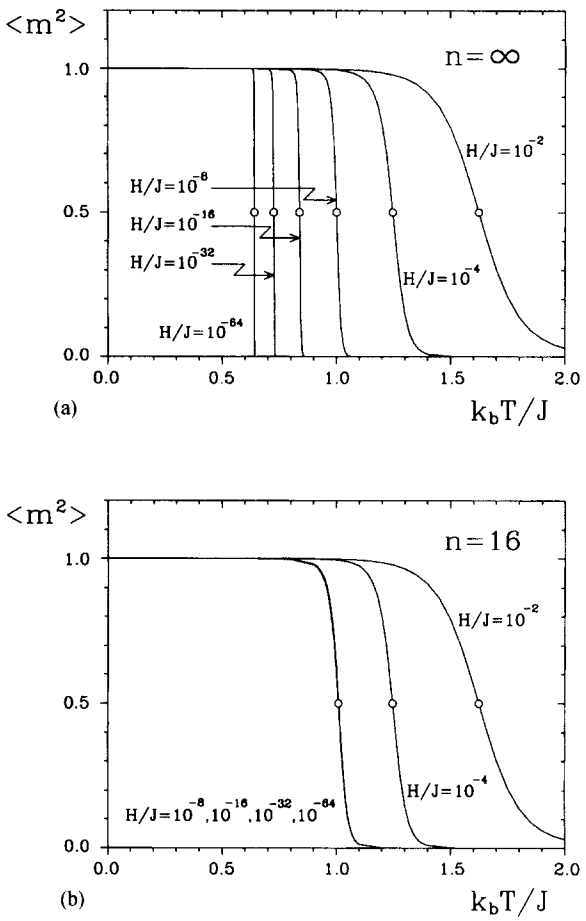

Fig. 2. Temperature dependence of the order parameter $\langle m^2 \rangle$ for different field values $H/J = 10^{-2}$, 10^{-4}, 10^{-8}, 10^{-16}, 10^{-32}, 10^{-64} for (a) $n = \infty$ (that is, n large enough so that the curves stop changing with a further increase of n), and (b) $n = 16$. Finite size ($n = 16$ in (b)) inhibits the shift towards zero temperature, with decreasing field, of the ordered phase region. Finite field also induces an apparent phase transition, vanishing only in the true limit of zero field.

where $h = H/J$ is the dimensionless field variable, $D = \ln 3/\ln 2$ is the fractal dimension of the gasket, and ξ is the correlation length which was shown [1,2] to have low temperature behavior,

$$\xi \sim \exp\left[\log 2\left(\frac{e^{4\beta J}}{4} + 4\beta J + e^{-4\beta J} - \frac{10}{3}e^{-8\beta J} + \cdots\right)\right]. \tag{7}$$

In zero field, curves $m(L,T)$ should collapse onto a single curve if plotted versus L/ξ, and from the definition of T_m it follows that L/ξ_m should be preserved, leading to limiting relation (5). In Fig. 3(a) we present numerical data for $T_m(L, H = 0)$ versus $\log(\log(L))$, corresponding to results of Ref. [3]. It is seen that even for the largest L

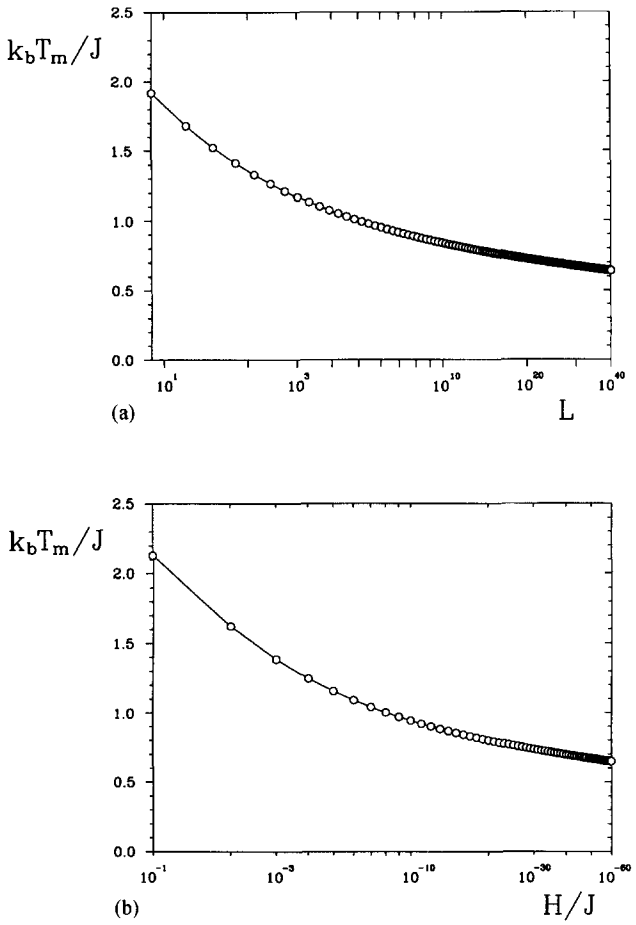

Fig. 3. Apparent magnetic transition temperature T_m versus (a) size (for zero field), and versus (b) field (for system sizes for which $\langle m^2 \rangle$ temperature dependence stops changing with a further increase of size). Note the double logarithmic scale on both graphs. Results for zero field were first obtained in Ref. [3], where it was shown that the thermodynamic limit presents a poor approximation for the Ising model on the Sierpiński gasket. From (b), a similar conclusion can be drawn for the limit $H \to 0$.

considered ($L = 10^{40}a \sim 10^{32}$ cm) equation (5) is still not a very good approximation for the behavior of T_m. This corresponds to the fact that at $k_BT_m/J \sim 0.65$, the first correction term in (7) still amounts to roughly 5% of the leading term. If we now first take the limit $L \to \infty$, we conclude that quantity $\xi_m^D h$ should be constant, leading to the analog of (5),

$$T_m(L = \infty, H \to 0^+) \sim \frac{1}{\ln \ln h^{-D}} . \tag{8}$$

In Fig. 3(b) we plot $T_m(L = \infty, H)$ versus $\log(\log(h^{-1}))$ (obtained for values of L for which T_m stops changing with further increase of L). From Fig. 3(b) it follows

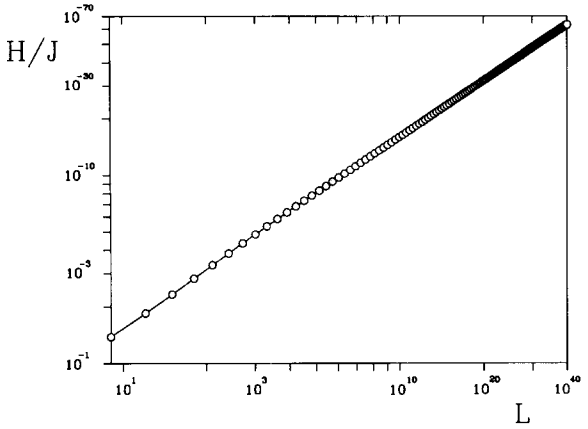

Fig. 4. Pairs of values $\log(\log(h^{-1}))$ and $\log(\log(L))$ that correspond to the same values of T_m in Fig. 2, representing the competition between finite size scaling and scaling with the field. The region above the line corresponds to dominance of finite size, and the region below the line to dominance of the field.

that even for inconceivably small fields of the order of $H/J \sim 10^{-60}$ (corresponding to 2×10^{-54} G for $J/k_B = 300$ K), $k_B T_m/J$ remains of the order of unity. As in the case $H = 0$, $L \to \infty$, we see that (8) is still not a very good approximation for the behavior of T_m, and that correction terms should be retained in (7). Nevertheless, Fig. 3 serves the purpose of drawing the analogy between the thermodynamic limit and the limit $H \to 0$, by displaying the fact that both limits present a poor approximation for the Ising model on the Sierpiński gasket.

Finally, to compare the influence of L and H on the behavior of T_m, in Fig. 4 we plot pairs of values $\log(\log(h^{-1}))$ and $\log(\log(L))$ that correspond to the same values of T_m in Fig. 3. The observed linear behavior in the whole region follows from equating the scaling variables L/ξ_m and $h^{-1/D}/\xi_m$. The region above the line corresponds to the dominance of finite size scaling ($\langle m^2 \rangle$ changes with L and is insensitive to small variations in H), and the region below the line corresponds to dominance of thermodynamic scaling ($\langle m^2 \rangle$ changes with H and is independent of L).

In conclusion, the extremely strong temperature dependence of the correlation function of the Ising model on the Sierpiński gasket gives rise to nontrivial thermodynamic behavior not only for large (but finite) systems, but also for infinite system in (tiny) finite fields. Both finite size and finite field induce an apparent phase transition, vanishing only in the true thermodynamic limit and the true limit of zero field, and both limits present poor approximations of realistic laboratory conditions. While the geometric properties of stochastic fractals have been widely studied over the past decade (see e.g. [6]), much less is known about the thermodynamic properties of model systems on fractal substrates. It remains to be shown whether such slow decay of correlations as given by (7) is encountered in real systems with fractal geometry, and if so, to which extent are finite size and finite field, encountered in real laboratory conditions, relevant for thermodynamic behavior of fractal magnetic samples.

Acknowledgements

This work was partially supported by CNPq (Brazilian Agency), the NSF, the Yugoslav–USA Joint Scientific Board under project JF900 (NSF), and the Serbian Science Foundation under project 0103.

References

[1] Y. Gefen, A. Aharony, Y. Shapir and B.B. Mandelbrot, J. Phys. A 17 (1984) 435.
[2] J.H. Luscombe and R.C. Desai, Phys. Rev. B 32 (1985) 1614.
[3] S.H. Liu, Phys. Rev. B 32 (1985) 5804.
[4] A.E. Ferdinand and M.E. Fisher, Phys. Rev. 185 (1969) 832.
[5] T. Stošić, B. Stošić, S. Milošević and H.E. Stanley, Phys. Rev. E 49 (1994) 1009.
[6] J. Feder, Fractals (Plenum Press, New York, 1988).