# Crossover from fractal lattice to Euclidean lattice for the residual entropy of an Ising antiferromagnet in maximum critical field $H_c$

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How thermodynamic properties of fractal objects cross over to the corresponding thermodynamic properties of nonfractal "Euclidean" objects is an open question of considerable recent interest. We study how the ground-state entropy $\sigma$ of the Ising antiferromagnet on a family of two-dimensional fractals "crosses over" to the ground-state entropy of a triangular lattice. The fractal family studied is a generalization of the simple Sierpiński gasket. We find that $\sigma$ varies smoothly with a parameter $b$ (which labels each member of the fractal family) and approaches for large $b$ the value $\sigma_{\text{Baxter}}=0.333\,242\,72\ldots$ calculated by Baxter and Tsang for the hard-hexagon problem on the triangular lattice and confirmed by Baxter to be an exact value.

## I. INTRODUCTION

How the laws of physics are modified when the substrate is fractal instead of Euclidean is a topic of tremendous current interest. Far less is known about the related question of how physical laws on fractal substrates "cross over" to know laws on translationally invariant Euclidean lattices. For example, the density of states $\rho(\omega)$ varies as $\omega^{d_s-1}$, where $d_s$ is the spectral dimension. Recently Borjan *et al.*[1] calculated $d_s$ exactly for a sequence of fractal objects that generalize the Sierpiński gasket. They found that the difference between the exact value $d_s=2$ for a Euclidean lattice and the exact values for $d_s$ for the fractal family is asymptotically a logarithimic function of $b$. Here $b=2,3,\ldots$ indexes the fractal objects in such a fashion that $b=2$ is the Sierpiński gasket and $b=\infty$ is a wedge of the triangular lattice.

The ground states of the Ising antiferromagnet have been studied by many authors. For example, Brooks and Domb[2] considered the square lattice with antiferromagnetic nearest-neighbor (NN) coupling $J$. They noted that when the magnetic field $H$ is decreased to a critical value $H_c=4J$, suddenly there is more than a single configuration with lowest energy. Hence there should be a nonzero entropy at the absolute-zero temperature $T$, and the Nernst theorem or the "third law" of thermodynamical should fail. More recently, such nonzero ground-state entropies have attracted attention in connection with their relevance for models of spin glasses.[3]

Most of the studies are concerned with the Ising model situated on translationally invariant lattices. However, it is becoming clear that there are many objects in nature that can be modeled by fractal lattices. Fractal lattices lack translation invariance but are characterized by dilation invariance. Thus it should be interesting to find ground-state properties of the Ising model situated on fractal lattices and to study the detailed relations that exist between a given physical quantity on a fractal lattice and the same quantity on a regular Euclidean lattice. One step in this direction is to study the subtle "crossover" from a fractal to a Euclidean lattice. To this end, we here study ground-state degeneracy of the Ising antiferromagnet situated on a family of fractal lattices.[4] The first member ($b=2$) of the family is the two-dimensional Sierpiński gasket while the last member ($b=\infty$) is a "60° wedge" of the ordinary triangular lattice. Each member of the family can be generated by a generator $G(b)$, where $b$ is an integer that runs from 2 to infinity.[4] Each $G(b)$ is an equilateral triangle (Fig. 1) that contains $b^2$ identical smaller triangles of unit side length, of which only the upward oriented are physically present. The fractal lattice is generated in the limit $n\rightarrow\infty$ of an iterative process shown in Fig. 1. Stage 2 is obtained by enlarging the generator ("stage 1") by a factor $b$ in linear dimension, filling the upward-pointing triangles with the stage-1 lattice and leaving the downward triangles empty. Stage $(n+1)$ is created from stage $n$ in the same fashion. Growing a fractal this way assures its invariance under scale transformations or "self-similarity." The fractal di-

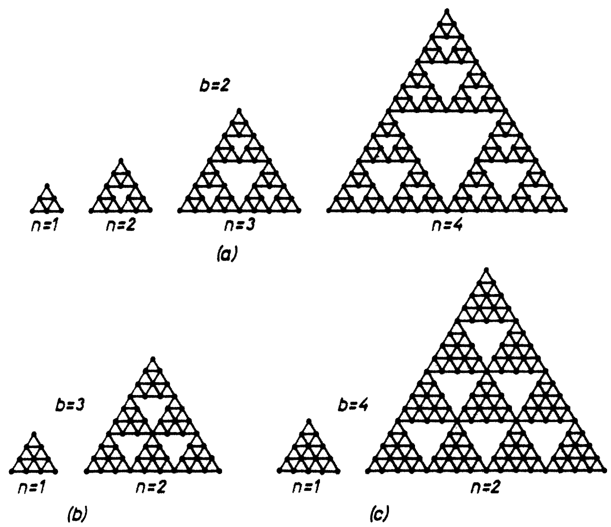


FIG. 1. Growth of the fractal lattice with (a) $b=2$ (the Sierpiński gasket), (b) $b=3$, and (c) $b=4$. The first stage ($n=1$) is termed the generator and it is designated $G(2)$. The complete fractal lattice is obtained in the limit $n \to \infty$. For $b>2$, there are two kinds of sites (●); some have four nearest neighbors and others have six.

mension $d_f$ depends monotonically on $b$, and is seen by inspection from Fig. 1 to be simply

$$d_f = \ln[b(b+1)/2]/\ln b \ . \tag{1}$$

Note from (1) that $d_f$ crosses over from its value for a fractal object to its value of 2 for a triangular wedge with the form $d_f = 2 - A/\ln b$, where $A = \ln 2$.

We focus on the ground-state degeneracy in the maximum critical field $H_c$ of an Ising antiferromagnet. The Hamiltonian is

$$\mathcal{H} = J \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i \ . \tag{2}$$

The first sum is over all NN pairs of spins; $s_i, s_i$ are the conventional Ising-spin variables ($s_i = \pm 1$) interacting with a positive coupling constant $J>0$ in a magnetic field $H$. Since $H_c$ depends on the number of nearest neighbors that a site of the lattice under study may have (see, for instance, the work of Hajdković and Milošević[5]), the first member of our class differs from all the other members. As can be seen in Fig. 1(a), each site of the Sierpiński gasket ($b=2$) has four nearest neighbors (except for the three "apex sites," which always have two nearest neighbors), whereas in the case of fractals with $b \geq 3$ some sites have six and some have four nearest neighbors [see Fig. 1(b)]. Consequently $H_c = 4J$ and $6J$ in the case $b=2$ and $b \geq 3$, respectively.

For values of $H$ larger than the critical value $H_c$, the system orders *ferromagnetically* at temperature $T=0$. At $H_c$, the ground-state energy of a system is highly degenerate in such a way that a large number of spins can be oriented against the field, provided their nearest neighbors remain parallel to the field (in the case $b \geq 3$ only spins with six nearest neighbors can be flipped, or otherwise the ground-state energy would increase). The degeneracy of the ground states is accompanied by nonzero residual entropies. It is the aim of the present paper to calculate and study these entropies.

In Sec. II we present our findings for the $b=2$ case. In Sec. III we analyze the case of fractals with $b \geq 3$, while in Secs. IV and V we present a summary discussion of the results and pertinent conclusions.

## II. THE $b=2$ CASE (SIERPIŃSKI GASKET)

### A. The maximum critical field $H_c$

Consider the generator of the Sierpiński gasket (Fig. 2). There are six spins, whose orientations we denote by $+1$ (spin up) and $-1$ (spin down). If $H=\infty$, then the lowest-energy configuration or "ground state" is the configuration with all six spins oriented up [(Fig. 2(a)]. Since $s_i = +1$ for $i=1,2,\ldots,6$ and there are nine nm pairs of spins, from (2) we see that its energy is

$$E = +9J - 6H \ . \tag{3a}$$

Now decrease $H$ from $\infty$. The lowest-energy configuration remains the same until $H$ reaches a critical value $H_c$ for which the three configurations of Fig. 2(b) have the same energy. The energy of these three configurations is

$$E = (5-4)J - (5-1)H_c \ . \tag{3b}$$

From comparison of (3a) and (3b) we see that if $H_c = 4J$ then the energy of all four configurations is $-15J$, so that we may that this energy level is the fourfold degenerate ground state of the $b=2$ generator. This will lead to a macroscopic ground-state degeneracy of the entire fractal object.

### B. Renormalization from stage $n$ to stage $n+1$

The key feature of the family of fractals studied in this paper is that it remains only to discover the rule by which the entropy of stage $n=2$ arises from that of stage $n=1$. Then since we have a self-similar exact fractal, we know the general rule whereby stage $n+1$ arises from stage $n$, and the problem is solved in general. For the sake of specificity, we derive this transformation from $n=1$ to $n=2$. The general case $n$ to $(n+1)$ is then the same. We begin by introducing four quantities $\Omega_i$ that are the degeneracies of the $n=1$ case when its apex spins are in

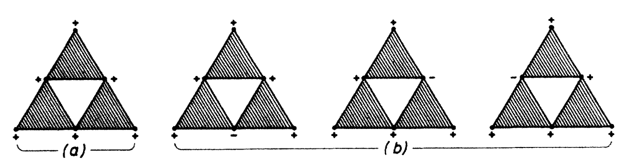


FIG. 2. (a) $T=0$ ground-state configuration of a stage-$n=1$ Sierpiński gasket for magnetic field $H$ above the critical field $H_c$. (b) The additional three configurations that become degenerate in ground-state energy with the configuration of (a) when $H=H_c=4J$.

one of the four possible configurations. Thus we define

$$\Omega_1 \equiv \text{degeneracy for apex configuration } \{+++\} , \tag{4a}$$

$$\Omega_2 \equiv \text{degeneracy for apex configuration } \{++-\} , \tag{4b}$$

$$\Omega_3 \equiv \text{degeneracy for apex configuration } \{+--\} , \tag{4c}$$

$$\Omega_4 \equiv \text{degeneracy for apex configuration } \{---\} . \tag{4d}$$

Our goal is to calculate the relations between $\Omega_i$ and the corresponding quantities $\Omega_i'$ for the stage $n=2$. To accomplish this systematically, we refer to Fig. 3, which shows the four configurations of apex spins of the stage $n=2$. For each of these four configurations there are $2^3=8$ configurations of the three spins that form the corners of the downward-oriented empty triangle. Thus Fig. 3(a) shows the eight configurations of "corner spins" corresponding to the configuration $\{+++\}$ of apex spins. Clearly the degeneracy $\Omega_1'$ is the sum of the degeneracies of each of these eight configurations.

From the definition (4a) we see that the first configuration contributes a degeneracy $\Omega_1^3$ since all three corner spins are $+1$. Similarly, from the definitions (4a) and (4b) we see that the next 3 configurations of Fig. 3(a) each contribute a term $\Omega_1\Omega_2^2$. From (4b) and (4c) the next three configurations contribute $\Omega_2^2\Omega_3$ while the last of the eight configurations of Fig. 3(a) contributes a term $\Omega_3^3$. Thus the total degeneracy of all possible configurations of the $n=2$ stage with apex spins all oriented upward is simply

$$\Omega_1' = \Omega_1^3 + 3\Omega_1\Omega_2^2 + 3\Omega_2^2\Omega_3 + \Omega_3^3 . \tag{5a}$$

Figure 3(b) shows the eight configurations of corner spins contributing to the degeneracy of the stage $n=2$ when the apex spins are in the configuration $\{++-\}$. From Fig. 3(b)–3(d) we see that

$$\Omega_2' = \Omega_1^2\Omega_2 + 2\Omega_1\Omega_2\Omega_3 + 2\Omega_2\Omega_3^2 + \Omega_2^2\Omega_4 + \Omega_3^2\Omega_4 + \Omega_2^3 , \tag{5b}$$

$$\Omega_3' = \Omega_1\Omega_2^2 + 2\Omega_2^2\Omega_3 + 2\Omega_2\Omega_3\Omega_4 + \Omega_1\Omega_3^2 + \Omega_3\Omega_4^2 + \Omega_3^3 , \tag{5c}$$

$$\Omega_4' = \Omega_2^3 + 3\Omega_2\Omega_3^2 + 3\Omega_3^2\Omega_4 + \Omega_4^3 . \tag{5d}$$

Note that the sum of the coefficients in each of the relations (5) is equal to 8. The degeneracy relations (5) have the same structure as the renormalization-group equations for the conditional partition functions derived by Luscombe and Desai.[6] This is a consequence of the self-similar structure of the Sierpiński gasket.

The recursion relations (5) can be applied iteratively since the same relations apply to transformation from any stage $n$ to the next stage $n+1$. The dimensionless residual entropy per spin $\sigma(b)$ is defined, for $b=2$, by

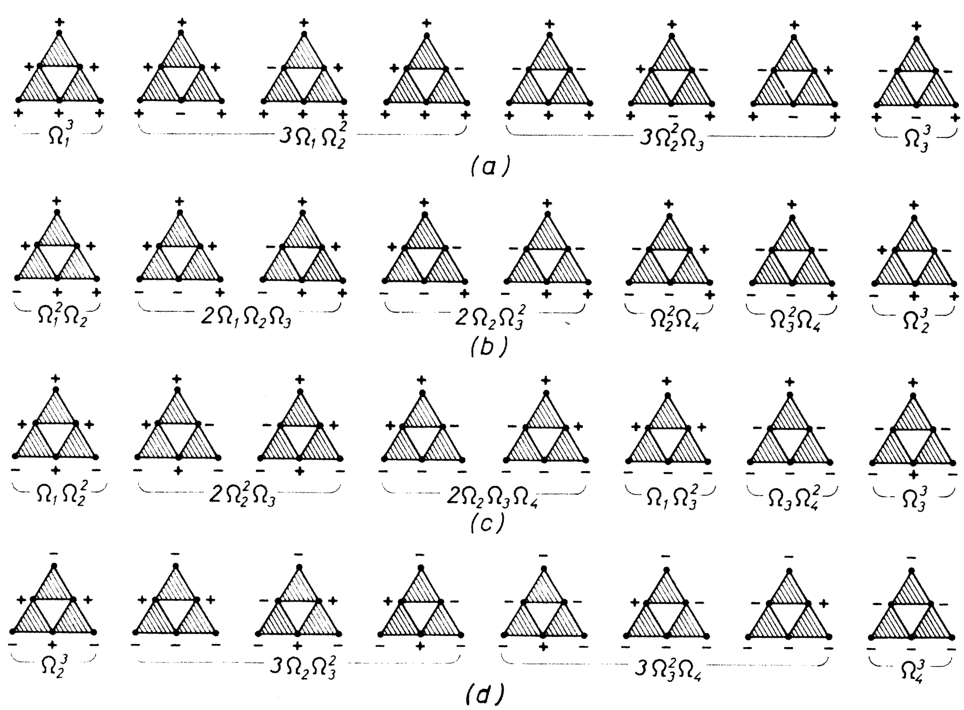

FIG. 3. Stage-$n=2$ finite-size fractal lattice ($b=2$) with the apex spins being fixed, whereas the three apex spins of the three interior stage-$n=1$ finite-size lattices assume all possible states; calculation of (a) $\Omega_1'$, (b) $\Omega_2'$, (c) $\Omega_3'$, and (d) $\Omega_4'$ given by Eqs. (5).

$$\sigma(2) \equiv \lim_{n\to\infty} \left[ \frac{\ln \Omega_i}{N_n} \right] . \tag{6}$$

Here $i$ can be *any* of the four integers (1,2,3,4) since the difference between the four quantities $\Omega_1$, $\Omega_2$, $\Omega_3$, and $\Omega_4$, scaled by the number of spins $N_n \equiv (3^{n+1}+3)/2$, approaches zero when $n \to \infty$. To test this idea, let us start from the initial conditions for the generator (stage $n=1$),

$$\Omega_1(n=1)=4, \quad \Omega_2(n=1)=2 , \\ \Omega_3(n=1)=1, \quad \Omega_4(n=1)=1 . \tag{7}$$

Equation (7) follows by inspection from Fig. 1. We find, after 18 iterative applications of (5) that,

$$\sigma(2)=0.384\,309\,53 \cdots \tag{8}$$

regardless of which of the four quantities $(\Omega_1,\Omega_2,\Omega_3,\Omega_4)$ we use in evaluating $\sigma(2)$. If we want to refine $\sigma(2)$ to a higher accuracy, we should perform further iterations of (5). In other words, the eight numerals given in (8) remain unchanged after the 18th iteration.

Note that the value found for $\sigma(2)$ lies between the lower bound $\sigma_l=0.321\,887\,6$ and the upper bound $\sigma_u=0.402\,359\,5$ predicted by Hajduković and Milošević[5] for systems with coordination number $z=4$. For $b\geq 3$, we shall see that the residual entropies $\sigma(b)$ will, up to a certain large value of $b$, remain below the lower bound $\sigma_l=0.297\,063\,1$ predicted[5] for systems with $z=6$. This should not be surprising since the fractals with $b\geq 3$ have an appreciable number of sites with $z=4$ (besides those with $z=6$), which are such that spins situated on them are constrained to stay parallel to the critical field ($H_c=6J$). Because they have two different coordination numbers, fractals with $b\geq 3$ have a smaller degree of the ground-state degeneracy in the maximum critical field.

## III. THE CASE $b \geq 3$

We have already noticed in Secs. I and II that the maximum critical field for fractals with $b\geq 3$ is $H_c=6J$ and that the corresponding ground-state degeneracy stems from allowing an arbitrary number of spins with six nearest neighbors to be either parallel or antiparallel to the field (while their nearest neighbors must stay parallel to the field). In a generator $G(b)$ there can be altogether

$$B=\frac{(b-2)(b-1)}{2} \tag{9}$$

spins which are so positioned that each of them has six nearest neighbors. Henceforth we shall call them the bulk spins. It is important to observe that spins at apexes of a finite-size fractal lattice and their nearest neighbors, as well as the spins on the edges of the lattice, are not the bulk spins [Figs. 1(b) and 1(c)]. Thus the edge spins stay parallel to the field and thereby the ground-state degeneracy of a finite-size structure can be represented by a single quantity (instead of the four that were used in the case $b=2$). Let $\Omega$ and $\Omega'$ be degrees of degeneracy at stage $n$ and $(n+1)$, respectively. Our goal is to find a relation between $\Omega$ and $\Omega'$.

When the stage-$n+1$ finite-size lattice is formed out of the stage-$n$ lattices, the apex spins of the latter become surrounded by the six nearest neighbors that have been parallel to the field. Accordingly, the stage-$n$ apex spins can be now arbitrarily oriented and since there are precisely $B$ of them [within the stage-$(n+1)$ structure] we can write the following relation:

$$\Omega'=2^B\Omega^C , \tag{10}$$

where

$$C=\frac{b(b+1)}{2} \tag{11}$$

is the number of the stage-$n$ lattices that comprise the stage-$n+1$ lattice. The relation (10) can be applied iteratively. Starting with the ground-state degeneracy $\Omega_G$ of the generator $G(b)$, we obtain the new relation

$$\Omega=\frac{(2^{B/(C-1)}\Omega_G)^{C^{n-1}}}{2^{B/(C-1)}} , \tag{12}$$

which is sufficient to calculate the corresponding residual entropy $\sigma(b)$ per spin, providing we know $\Omega_G$. In fact, leaving aside for a moment the question of $\Omega_G$, we can adapt (6) by substituting (12) for $\Omega_i$. The total number of spins of the stage-$n$ structure $N_n$ is somewhat more complex in the case $b\geq 3$ than in the case $b=2$. Yet one can readily check that for $b\geq 3$,

$$N_n=C^{n-1}N_G-\frac{[3(b-1)+2B](C^{n-1}-1)}{C-1} , \tag{13}$$

where $N_G$ is the number of spins that can be situated on the generator $G(b)$,

$$N_G=\frac{(b+1)(b+2)}{2} . \tag{14}$$

Therefore, inserting (12), (13), and (14) into (6), we obtain, after a straightforward calculation, the final expression for the residual entropy,

$$\sigma(b)=\frac{2\{(b-1)(b-2)\ln 2+[b(b+1)-2]\ln\Omega_G\}}{b(b^2-1)(b+4)} . \tag{15}$$

We see that due to the self-similarity of the fractals, the residual entropy, per spin, of the infinite lattice is a simple function of the ground-state degeneracy $\Omega_G$ of the corresponding generator. Hence our next step consists in evaluating $\Omega_G$ for various $b$.

For small $b$, up to $b=7$, one can calculate $\Omega_G$ straightforwardly. However, for larger $b$ the calculation becomes laborious and one should use a computer. The calculation of $\Omega_G$ is actually a problem of determining the number of possible configurations of $B$ bulk spins that satisfy the condition that no two of them, if they are nearest neighbors, can be simultaneously antiparallel to the field. A simple FORTAN program that surveys all possible configurations and determines $\Omega_G$, for small $b$, can be easily written but the problem because formidable for large $b$ since the number of all configurations of $B$ spins is equal to $2^B$, where $B$ is given by (9). For $b=11$ there are $2^{45}\approx 3\times 10^3$ configurations and this is very close to the upper operational limit of the present-day computers. In order to surmount the computational difficulties we have applied the method indicated by Binder,[7] in a somewhat different form.

The outline of our computational method is briefly described as follows. For each generator $G(b)$ we consider the chain of $b-2$ bulk spins that is parallel to one of the sides of the triangle (Fig. 4). These spins may have $m=2^{b-2}$ configurations altogether. Let $\gamma_b(\lambda)$ be the ground-state degeneracy of the generator associated with one of the configurations, designated as $\lambda$, of the chain. The ground state degeneracy of $G(b)$ is then

$$\Omega_G=\sum_{\lambda=1}^{m}\gamma_b(\lambda) . \qquad (16)$$

Thus knowing all configurations of the chain of bulk spins and the corresponding quantities $\gamma_b(\lambda)$ we know the ground-state degeneracy of the generator $G(b)$. But the virtue of this knowledge appears in the next step. In order to determine $\Omega_G$ of $G(b+1)$ we consider the new chain of $b-1$ spins that is parallel to the foregoing chain. The $2^{b-1}$ configurations of the new chain we shall label by $\lambda'$. Defining a projector $\hat{P}$ by setting $P_{\lambda\lambda'}=1$ if the configurations $\lambda$ and $\lambda'$, of the $b-2$ and $b-1$ chains, respectively, are compatible (in the sense that no two neighboring spins of theirs' should be simultaneously antiparallel to the field), and $P_{\lambda\lambda'}=0$ otherwise, we may write

$$\gamma_{b+1}(\lambda')=\sum_{\lambda=1}^{m}P_{\lambda\lambda'}\gamma_b(\lambda) . \qquad (17)$$

Hence it follows that by comparing the configurations of two adjacent chains of bulk spins we can learn the ground-state degeneracies of the generators. In this way the computational effort needed to evaluate $\Omega_G$ amounts to approximately $2^{b-3}\times 2^{b-2}$ comparisons, which is, for large $b$, many orders of magnitude less than a straightforward survey of all possible configurations of $G(b)$ (for example, for $b=11$ instead, to analyze about $3\times 10^{13}$ configurations one has to make $2^{17}\approx 10^5$ comparisons). In fact, by applying the method just explained, and with some additional improvements, we have been able to extend, and confirm, our straightforward calculations of $\Omega_G$ (implemented for $b\leq 9$) up to $b\leq 21$.

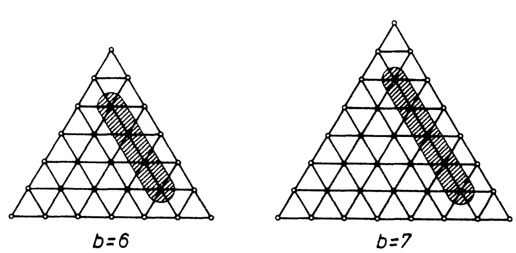


FIG. 4. Generators $G(6)$ and $G(7)$ of the $b=6$ and $b=7$ fractals. The shaded regions contain two successive chains of sites such that each site has six nearest neighbors. Spins positioned at these sites (●) are termed the bulk spins.

## IV. RESULTS

In Table I we present the residual entropies of the 19 members of the class of fractals that has been studied in this paper. For the sake of comparison, we present also residual entropies of the corresponding generators calculated according to the formula

$$\sigma(b)^{(1)}=\frac{\ln\Omega_G}{N_G} , \qquad (18)$$

where $\Omega_G$ is the ground-state degeneracy of the generators and $N_G$ is given by (14). It is evident that as $b$ increases the difference between the two sets of data decreases. This should have been expected since when $b\to\infty$ the generators are approaching, for obvious reasons, the ordinary triangular lattice, whereas in the case of the corresponding fractals, judging according to the established limit of their fractal dimension[4]

$$d_f\to 2, \quad b\to\infty , \qquad (19)$$

we may say that they also converge to the two-dimensional Euclidean lattice. Accepting the latter statement, one may argue that the pertinent residual entropies of both the fractals and the corresponding generators should, in the limit $b\to\infty$, approach the residual entropy of the Ising model situated on the ordinary triangular lattice, in the maximum critical field,

$$\sigma_{\text{Baxter}}=0.333\,242\,721\,976\,1\cdots . \qquad (20)$$

This value was calculated by Baxter and Tsang[8] and confirmed by Baxter[9] to be an exact value. In Fig. 5 we plot the residual entropies calculated in this paper against the $1/b$ values and we see that it is very plausible to expect that the calculated entropies converge to (20) when $b\to\infty$.

In Sec. I we raised the question of how the residual en-

TABLE I. The dimensionless residual entropies of the fractals and their generators, $\sigma(b)$ and $\sigma(b)^{(1)}$, respectively, in the maximum critical field $H_c=6J$.

| $b$ | $\sigma(b)$ | $\sigma(b)^{(1)}$ |
|---|---|---|
| 3 | 0.099 021 026 | 0.069 314 718 |
| 4 | 0.121 300 776 | 0.092 419 624 |
| 5 | 0.152 243 28 | 0.125 669 40 |
| 6 | 0.169 177 83 | 0.146 226 59 |
| 7 | 0.185 134 80 | 0.165 295 63 |
| 8 | 0.197 291 80 | 0.180 158 17 |
| 9 | 0.207 844 58 | 0.192 953 80 |
| 10 | 0.216 717 15 | 0.203 695 75 |
| 11 | 0.224 436 27 | 0.212 971 98 |
| 12 | 0.231 164 42 | 0.221 006 06 |
| 13 | 0.237 105 16 | 0.228 048 93 |
| 14 | 0.242 383 66 | 0.234 264 24 |
| 15 | 0.247 109 05 | 0.239 791 40 |
| 16 | 0.251 363 81 | 0.244 736 93 |
| 17 | 0.255 215 90 | 0.249 187 90 |
| 18 | 0.258 720 17 | 0.253 214 45 |
| 19 | 0.261 922 03 | 0.256 874 33 |
| 20 | 0.264 859 20 | 0.260 021 21 |
| 21 | 0.267 563 31 | 0.263 276 91 |

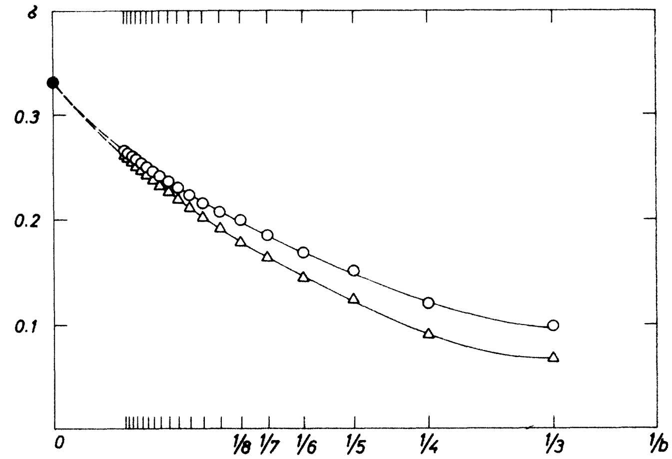


FIG. 5. Residual entropies of the fractals (○) and their generators (△) plotted against the variable $1/b$. The closed circle (●) at the vertical axis ($b=\infty$) corresponds to the residual entropy of the ordinary triangular lattice (Refs. 8 and 9). It should be noted that only for very large $b$ the residual entropies of the fractals surpass the lower limit $\sigma_l=0.297\,06\,31$ established (Ref. 5) for a lattice with the constant coordination number $z=6$. The solid and dashed lines serve as a guide to the eye.

tropies of fractals converge to the residual entropy of Euclidean lattice. This question is related to numerous recent questions (see, for example, Refs. 10 and 11) which are concerned with the way physical laws on fractals approach the known laws on ordinary translationally invariant lattices. For instance, it follows from (1) that the asymptotic form of the fractal dimension is $d_f=2-\ln 2/\ln b$ for very large $b$, and it was recently argued that the spectral dimension $d_s$ of the fractals under study also has a logarithmic asymptotic law.[1] Since the spectral dimension determines many dynamical properties of fractals (see, for instance, Ref. 12), one could assume that a logarithmic asymptotic law may be relevant in the case of residual entropies as well. For this reason we have performed the least-square fitting of our data for $\sigma(b)$ to the formula

$$\sigma(b)=\sum_{l=0}^{m} a_l\left[\frac{1}{x}\right]^l , \tag{21}$$

where $x$ has been consecutively assumed to be $b$, $\ln b$, and $b^\alpha$, while $a_l$ and $\alpha$ are the fitting constants.

Accepting at most four-fitting constants ($m=3$), we have found that is the last choice, $x=b^\alpha$, and not $x=\ln b$, which offers a fitting function (21) that has smallest deviations from $\sigma(b)$ given in Table I and, at the same time, gives the best reproduction of $\sigma(b)$ given by (20) (our findings are depicted in Fig. 6). In order to strengthen this conclusion we have carried out an additional least-square fitting to the formulas

$$\sigma(b)=\sigma_{\text{Baxter}}-\frac{P}{b^\alpha} \tag{22}$$

and

$$\sigma(b)=\sigma_{\text{Baxter}}-\frac{Q}{(\ln b)^\beta} , \tag{23}$$

TABLE II. The fitting constants $\alpha$, $\beta$, $P$, and $Q$ of functions (22) and (23) fitted to the residual entropies $\sigma(b)$ given in Table I. The sums of the squared deviations, that is the quantities

$$\sum_{b=22-N}^{21}[\sigma(b)(\text{exact})-\sigma(b)(\text{fitted})]^2 ,$$

are multiplied by $10^{10}$ and denoted by $D$, whereas $N$ is the number of data included in each fit.

| $N$ | $D$ | $\alpha$ | $P$ | $D$ | $\beta$ | $Q$ |
|---|---|---|---|---|---|---|
| 19 | 4 122 108 | 0.642 | 0.499 | 25 534 564 | 1.53 | 0.289 |
| 18 | 732 769 | 0.687 | 0.556 | 7 423 863 | 1.392 | 0.352 |
| 17 | 614 462 | 0.698 | 0.57 | 4 136 563 | 1.523 | 0.395 |
| 16 | 214 521 | 0.721 | 0.605 | 1 749 584 | 1.664 | 0.45 |
| 15 | 126 739 | 0.734 | 0.626 | 920 231 | 1.774 | 0.5 |
| 14 | 57 803 | 0.748 | 0.65 | 444 000 | 1.867 | 0.546 |
| 13 | 30 709 | 0.759 | 0.67 | 225 935 | 1.949 | 0.593 |
| 12 | 14 780 | 0.769 | 0.688 | 113 056 | 2.032 | 0.644 |
| 11 | 8248 | 0.778 | 0.705 | 55 454 | 2.092 | 0.686 |
| 10 | 3385 | 0.786 | 0.722 | 26 585 | 2.148 | 0.727 |
| 9 | 1847 | 0.791 | 0.733 | 12 788 | 2.202 | 0.77 |
| 8 | 1736 | 0.793 | 0.736 | 5659 | 2.25 | 0.81 |
| 7 | 1195 | 0.795 | 0.742 | 2228 | 2.3 | 0.854 |
| 6 | 389 | 0.803 | 0.758 | 833 | 2.345 | 0.896 |

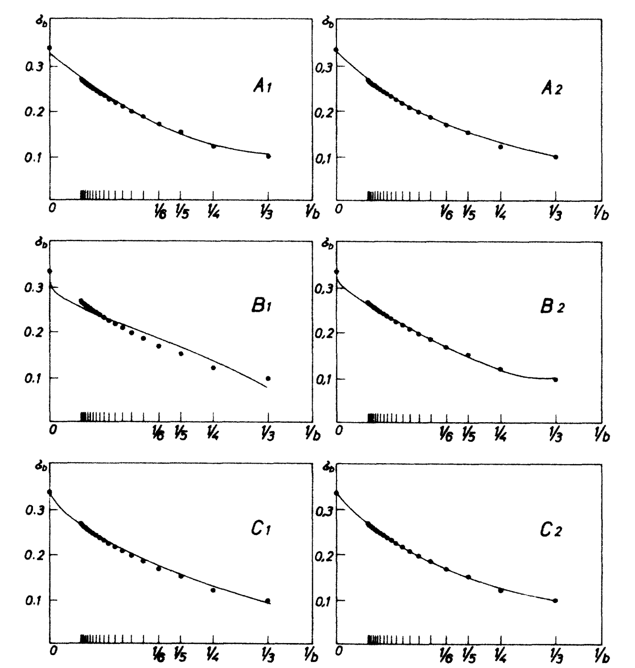


FIG. 6. Results of the least-square fitting to the data $\sigma(b)$, given in Table I, to function (21). Cases $A$, $B$, and $C$ correspond to the choices $x=b$, $x=\ln b$, and $x=b^\alpha$, respectively. The parameter $m$ was set equal to 1 (in the case $C_1$), 2 (in the case of $A_1$, $B_1$, and $C_2$), and finally 3 (in the case of $A_2$ and $B_2$). The solid lines represent function (21) with the optimized fitting constants whereas the data from Table I, and (20), are represented by (●).

where $P$, $Q$, $\alpha$, and $\beta$ are the fitting constants. This time we have varied the number of data included in the fit, and, again, we have found (Table II) that the logarithmic asymptotic formula (23) is less tenable than the similar power-law asymptotic formula (22). Thus we may argue that, according to our data, (22) is the limiting form of the residual entropies of the Ising antiferromagnets on the Sierpiński type of fractals with $b \geq 3$.

## V. CONCLUDING REMARKS

In summary, we have found the general formula (15) for the residual entropies of the Ising antiferromagnet, in the maximum critical field, situated on the Sierpiński-type family of fractals with $b \geq 3$. In addition we have calculated specific values of the residual entropies $\sigma(b)$ for $3 \leq b \leq 21$ and demonstrated that it is most likely that $\sigma(b)$ converge, when $b \rightarrow \infty$, to the value (20) calculated for the ordinary triangular lattice.[8,9] Numerical analysis of our findings reveals that the asymptotic law of $\sigma(b)$ cannot be of the logarithmic form, which, in conjunction with the fact that both the fractal $d_f$ and spectral dimension $d_s$ have logarithmic power laws,[1] implies that $\sigma(b)$ cannot be a simple function of $d_f$ and $d_s$. This result is in accord with the recent finding[13] that the critical exponents describing self-avoiding walks on the Sierpiński-type of fractals cannot also be simple functions of $d_f$ and $d_s$. Since the Ising magnets (with short-range interactions) on these fractals do not exhibit critical behavior, and consequently one cannot say anything about magnetic critical exponents, it would be interesting to study residual entropies of the Ising antiferromagnets on those fractals, for example on the Sierpińsi carpets,[14] for which it has been found that magnetic critical exponents are functions of $d_f$ and an additional geometric characteristic of fractals. This is a topic for further work.

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