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Distribution of shortest paths at percolation threshold: application to oil recovery with multiple wells

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Abstract

In this work we investigate the distribution of shortest paths in percolation systems at the percolation threshhold in two dimensions (2D). We study paths from one given point to multiple other points. In oil recovery terminology, the given single point can be mapped to an injection well (injector) and the multiple other points to production wells (producers). In the previously studied standard case of one injection well and one production well separated by Euclidean distance r, the distribution of shortest paths l, $P(l|r)$, shows a power-law behavior with exponent $g_l = 2.14$ in 2D. Here we analyze the situation of one injector and an array A of producers. Symmetric arrays of producers lead to one peak in the distribution $P(l|A)$, the probability that the shortest path between the injector and any of the producers is l, while the asymmetric configurations lead to several peaks in the distribution. We analyze configurations in which the injector is outside and inside the set of producers. The peak in $P(l|A)$ for the

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symmetric arrays decays faster than for the standard case. For very long paths all the studied arrays exhibit a power-law behavior with exponent $g \simeq g_l$.

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Keywords: Percolation; Oil recovery; Petroleum reservoirs; Multiple wells; Chemical path; Breakthrough time

1. Introduction

A common technique for oil recovery is fluid displacement. Liquid (e.g., water) or gas (e.g., carbon dioxide) is injected into one or more wells to displace the oil in the reservoir to one or more production wells. After some time, known as the breakthrough time, the injected fluid reaches a production well indicating that the usefullness of the production well is close to the end. For economic purposes, it is important to estimate the breakthrough time. Usually the oil industry does this estimation using a fluid mechanics simulation approach. A low cost, and fast alternative way to make a similar estimation is done using percolation theory [1–3]. Such estimation is performed based on the analysis of the shortest path (chemical distance) between two wells.

The chemical distance, l, between two sites is defined as the shortest path constructed over occupied sites or bonds, linking the two given sites. In the oil industry context this is the shortest path, inside the oil reservoir, between one injector and one producer. It is well known that the average shortest path, $\bar{l}$, scales as $r^{d_{\min}}$ where r is the Euclidean distance between the sites and the critical exponent $d_{\min} = 1.13$ [4,5] in two dimensions (2D). The quantity of interest here is the conditional probability, $P(l|r)$, that the shortest distance between two sites belonging to the same cluster, separated by Euclidean distance r, is l. $P(l|r)$ is roughly related to the breakthrough time from the injector to the producer [6–8].

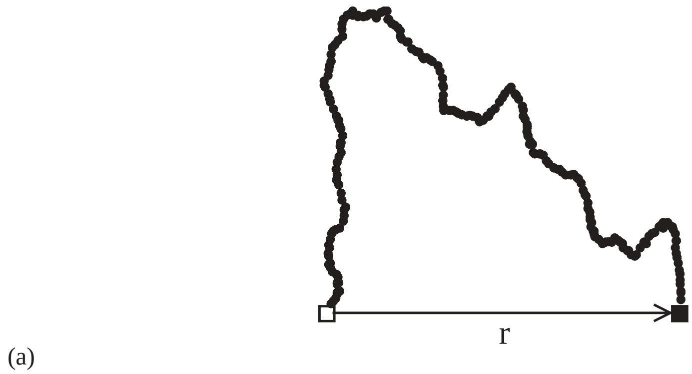
The basic situation with one injector and one producer is illustrated in Fig. 1. In Fig. 1(a) are shown two wells and a typical shortest path connecting these two. Fig. 1(b) is the log–log plot of shortest path distribution, $P(l|r)$, at the percolation threshold, versus $l/r^{d_{\min}}$. The shape of the plot of $P(l|r)$ in Fig. 1(b) is well known in the literature and is used in the estimation of breakthrough time and associated well management in oil industry [1–3]. The distribution $P(l|r)$ has one global maximum at

$$l \simeq r^{d_{\min}} . \tag{1}$$

We expect all characteristic lengths of the distribution to scale as in Eq. (1).

After the peak, the distribution $P(l|r)$ exhibits a power-law decrease. The exponent of the power-law, $g_l = 2.14$, is consistent with previous results. In the next sections we compare this plot with plots for more complex well configurations.

There are several studies [1,6,7] concerning the behavior of $P(l|r)$ for the case of one injector and one producer. In this work we investigate the effect of increasing the number of production wells. Specifically, we study $P(l|A)$ for the case of one injection well and an array A of producers each at the Euclidean distance r_i from the injection



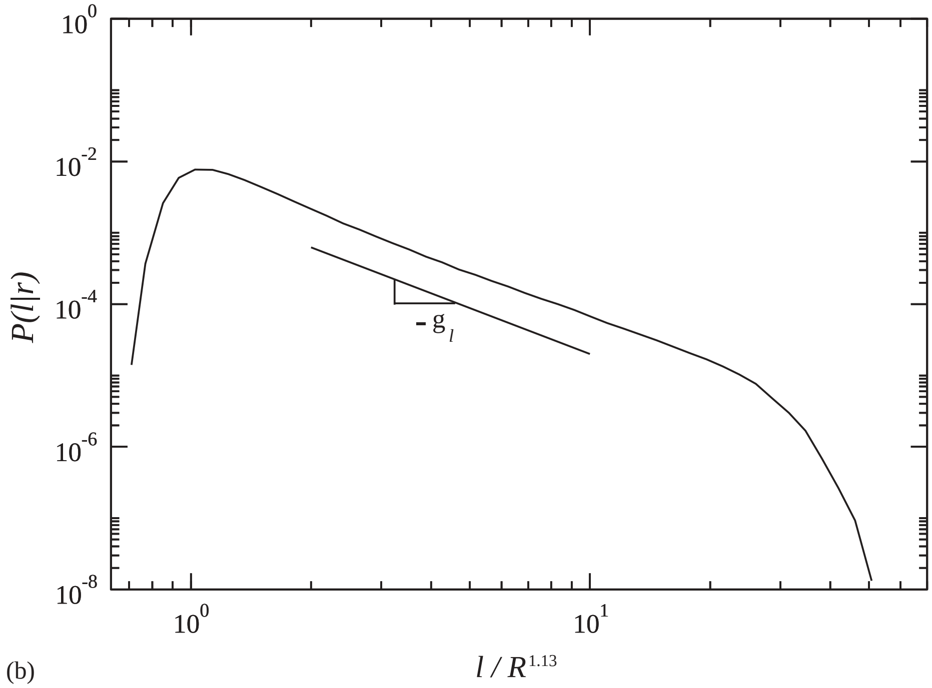


Fig. 1. The standard case of one injector and one producer. In (a) the layout of the array, the injector is indicated by an empty square, the producer by a filled square, the Euclidean distance between them by an arrow, and a typical path by a line. (b) The log–log plot of shortest path distribution, $P(l|r)$, at the percolation threshold.

well. The shortest path is now taken to be the shortest path from the injector well to any of the producer wells. We focus our analysis on configurations in which the producers lie on a circle of radius R. We analyze three distinct cases: the injector is at the center of the circle (symmetric case), the injector is inside the circle but not at the center (internal asymmetry), and the injector is outside the circle (external asymmetry).

All studies are done at the percolation threshold on a square lattice with side $L = 500$. Over 1 million realizations were performed for each configuration studied.

2. The symmetric case

This section focus on the case of one injector well and a symmetric array of producers. All the production wells are placed equidistant to the injector; the Euclidean distances between the injector and the producers are $r_i = R$. Fig. 2(a) illustrates the case of 4 producers. In the case of 2 producers all the wells are on a straight line.

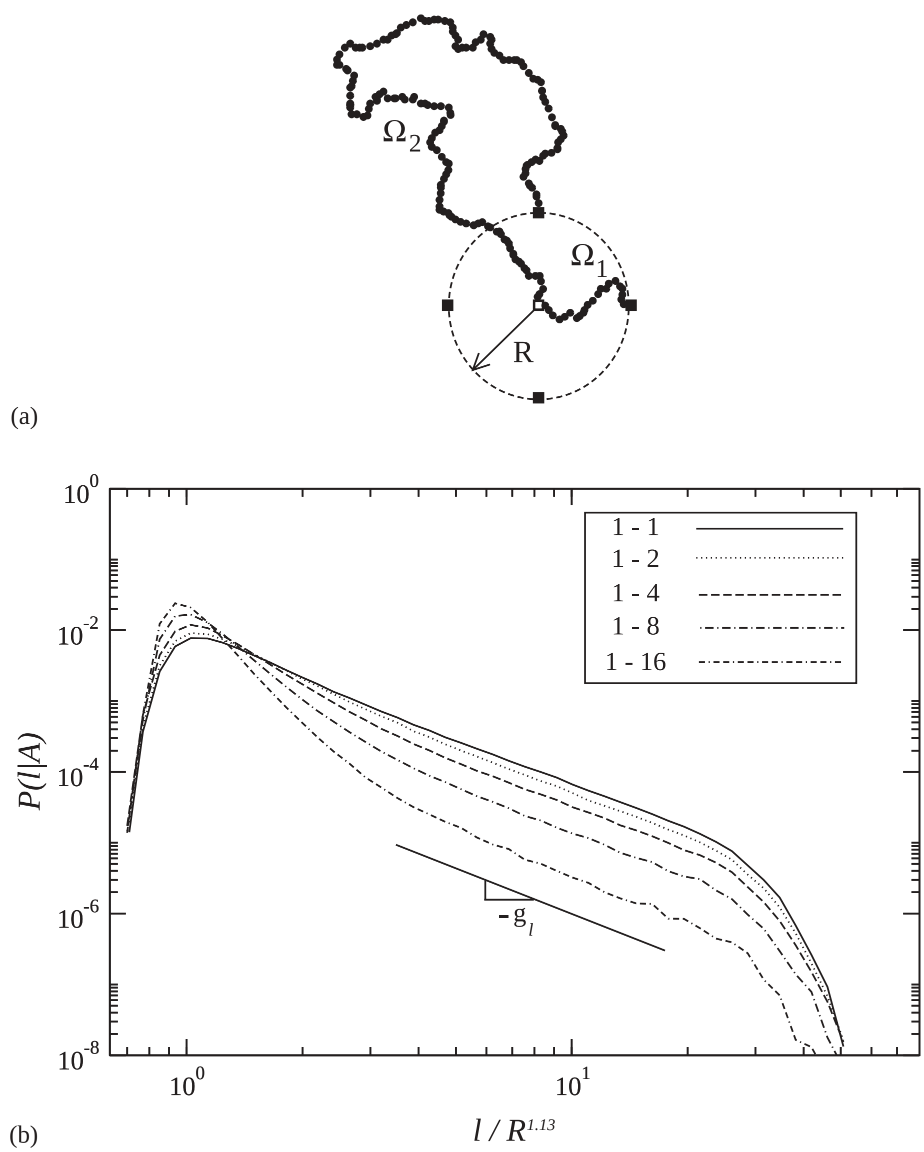


Fig. 2. The case of an injector at the center of a symmetric array of producers. (a) The layout of the array of multiple symmetric wells, the injector (empty square) is inside a circle of 4 producers (filled squares). The producers are in a circle (dashed line) of radius R. Two paths are indicated: Ω_1 is a short path which does not leave the circle and Ω_2 a very long one which does leave the circle. (b) Log–log plot of the shortest path distribution, $P(l|A)$, versus $l/R^{d_{\min}}$ at the percolation threshold. In the simulations $L = 500$ and $R = 64$. Five cases corresponding to 1, 2, 4, 8, and 16 production wells are indicated in the figure.

We also study cases in which the producers are at the vertices of a square, a regular octagon and a regular sixteen sides polygon; these are the cases of 4, 8 and 16 producers, respectively.

Fig. 2(b) is a log–log plot of the shortest path distribution, $P(l|A)$, versus $l/R^{d_{\min}}$, at the percolation threshold for the arrays of Fig. 2(a). Plots for five configurations are shown in Fig. 2(b): the standard configuration of one injector and one producer and configurations of 2, 4, 8 or 16 producers.

The plots of $P(l|A)$ exhibit a global maximum and upper and lower exponential cutoffs. We expect the maximum value of $P(l|A)$ for $(l/R^{d_{\min}})_{\max} \simeq 1$. The upper cutoff is due to the finite size of the system; the lower cutoff is due to the fact that the shortest path cannot be shorter than the Euclidean distance between the injector and the producer wells. The maximum of $P(l|A)$ is more accentuated for a large number of producers because the probability to reach a producer increases with their number. For the same reason, the maximum of $P(l|A)$ is driven slightly to the left in the figure as the number of producers is increased.

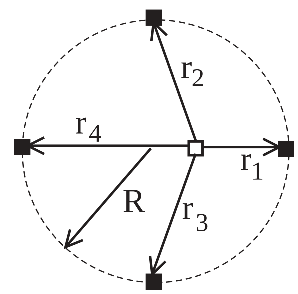
For a large number of producers the symmetric distribution $P(l|A)$ shows two regimes which can be characterized by the decay rates of the curves. For l just after the peak of $P(l|A)$ we have the first regime and for large l the second one. The first regime is characterized by an accentuated decaying rate. In this case the probability of short paths to reach a producer is greater than in the standard case of a single injector and producer; there is a competition between the producers to catch the paths originated from the injector. In the second regime we have a power-law with exponent $g \simeq g_l$. $P(l|A)$ shows a crossover at l_c the position of which depends on the number of producers. We explain this crossover with help of the following picture. There are two types of paths: paths which remain solely inside the circle of producers and paths which do not. We illustrate these two kind of paths with the schematic paths Ω_1 and Ω_2 in Fig. 2(a). The greater the number of producers, the greater the probability of a path of type Ω_1 which contributes to the peak of $P(l|A)$. The path of type Ω_2 is related to the power law in Fig. 2(b). For a large number of producers there is a very low probability that a path with origin at the injector escapes from the internal region; the probability of a path to reach at least one producer, before going outside the circle of producers, is very high. The escaping paths are rare but can be very long after escaping from the circle of producers. In some sense, these long paths see the set of producers as a single collecting spot. Thus for large l, this case is equivalent to the situation with one injection and one production well. In short, the physics behind the two regimes is distinct and implies different behavior. For the internal paths, $l < l_c$, there is a higher peak followed by a fast decay of $P(l|A)$ due to the strong encirclement and the almost complete capture of the paths by the producers. For the external paths, $l > l_c$, the set of producers behaves as a single average producer and as a consequence $g \simeq g_l$.

3. The asymmetric case

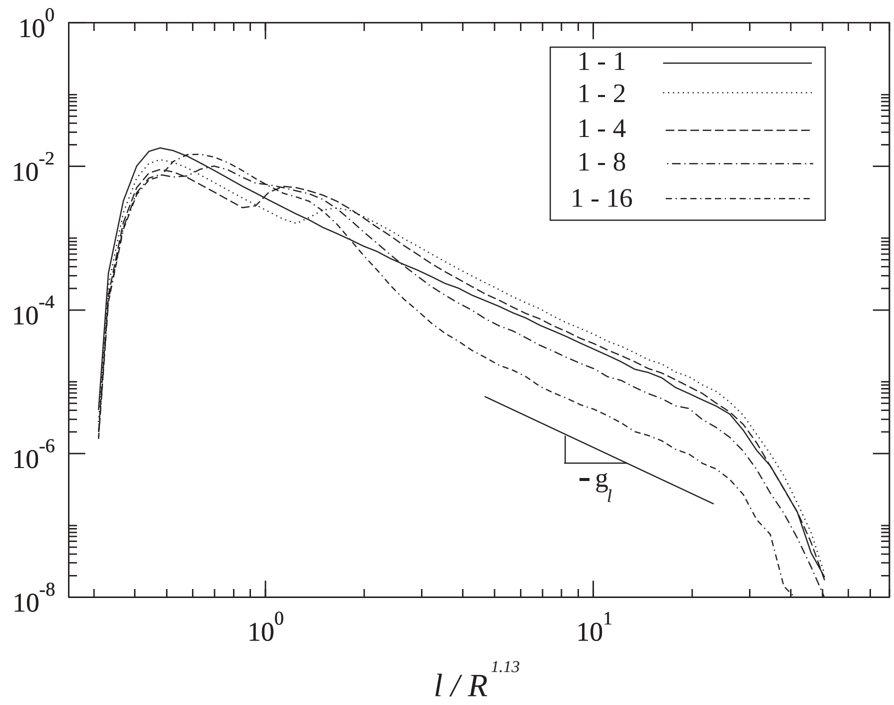
To gain an understanding of the distributions for generic situations we now study the configuration in which the injection well is not at the center of the circle of production wells and we compare our results with the symmetric case. The most important feature of the distributions $P(l|A)$ for the symmetric case is the presence of a single maximum. For the asymmetric case we will find multiple maxima.

3.1. Internal asymmetry

This subsection treats the case of one injector and an asymmetric array of producers where the injector is inside the circle of producers. Fig. 3(a) illustrates this configuration



(a)



(b)

Fig. 3. The case of an injector placed inside and asymmetrically to an array of producers. (a) The layout of the array of wells: the injector (empty square) is internal to a circle of 4 producers (filled squares). The producers are over a circle (dashed line) of radius R. The Euclidean distances r_i, $1 \leqslant i \leqslant 4$ are indicated by arrows. (b) Log–log plot of shortest path distribution, $P(l|A)$, versus $l/R^{d_{\min}}$ at the percolation threshold. In the simulations $L = 500$, $R = 64$ and $R_1 = 32$. Five cases corresponding to 1, 2, 4, 8, and 16 production wells are indicated in the figure.

for 4 producers. The producers are on a circle of radius $R = 64$. The injector, however, is not at the center of the regular polygons, but at distance $r = 32$ from the center. Thus, the distance from the injector to the closest producer is $r_1 = 32$ for all cases.

Fig. 3(b) contains log–log plots of the shortest path distribution, $P(l|A)$, versus $l/R^{d_{\min}}$ for the arrays of Fig. 3(a). Five plots are contained in Fig. 3(b): the standard case of one injector and one producer and configurations with 2, 4, 8 or 16 producers.

The case of one injector and two asymmetric producers shows a clear two maxima of $P(l|A)$ located at $(l/R^{d_{\min}})_{\max_1}$ and $(l/R^{d_{\min}})_{\max_2}$. The positions of two maxima scale as $(l/R^{d_{\min}})_{\max} \simeq (r_i/R)^{d_{\min}}$, for $i = 1, 2$. We see that $(l/R^{d_{\min}})_{\max_1} \simeq (32/64)^{1.13} \simeq 0.45$ and $(l/R^{d_{\min}})_{\max_2} \simeq (96/64)^{1.13} \simeq 1.6$. Similarly, the case of one injector and four producers should exhibit three maxima corresponding to r_1, $r_2 = r_3$ and r_4. We expect these maxima to be located at $(l/R^{d_{\min}})_{\max_1} \simeq (32/64)^{1.13} \simeq 0.45$, $(l/R^{d_{\min}})_{\max_2} \simeq (70/64)^{1.13} \simeq 1.1$ and $(l/R^{d_{\min}})_{\max_1} \simeq (96/64)^{1.13} \simeq 1.6$. We can identify the first maximum in the plot but the other two maxima overlap and cannot be resolved. They are not seen as separated peaks but instead as an integrated one. For the same reason the cases of eight and sixteen producers also exhibit roughly two peaks. What we have indeed are overlaping maxima with the first peak being progressively incorporated to the unresolved "mountain" of the distribution as the number of producers increases. It is good to point out that, in the plot, the separation between partial maxima decreases when the number of producers becomes large. The presence of several maxima in the asymmetric case also implies that the first maximum of $P(l|A)$ is less accentuated for a greater number of producers. For large l we see in Fig. 3(b) that the distributions follow a power law with $g \simeq g_l$. This regime, as in the symmetric case, corresponds to the long paths that have escaped from the internal region of producers.

3.2. External asymmetry

This subsection focuses on the case of one injector well which is outside the set of producers. Fig. 4(a) illustrates this situation for 4 producers. The producers lie on a circle radius $R = 16$. The injector is outside the circle at the distance $r = 32$ from to the center. The distance from the injector to the closest producer is thus $r_1 = 16$ for all the cases.

Fig. 4(b) is a log–log plot of the shortest path distribution, $P(l|A)$, versus $l/R^{d_{\min}}$ for the arrays of Fig. 4(a). The case of one injector and two producers clearly exhibits two maxima $P(l|A)$ located at $(l/R^{d_{\min}})_{\max_1}$ and $(l/R^{d_{\min}})_{\max_2}$. As before, the locations of these two maxima correspond to $(l/R^{d_{\min}})_{\max_i} \simeq (r_i/R)^{d_{\min}}$. We expect $(l/R^{d_{\min}})_{\max_1} \simeq (16/16)^{1.13} \simeq 1$ and $(l/R^{d_{\min}})_{\max_2} \simeq (48/16)^{1.13} \simeq 3.4$. We expect that for each distinct distance to the producer r_i to see a peak at $(l/R^{d_{\min}})_{\max} \simeq (r_i/R)^{d_{\min}}$, but as seen above some of the peaks cannot be discerned. As in the internal asymmetric case the amplitude of the first peak of $P(l|A)$ decreases with the number of producers. For large l we see in the curves of Fig. 4(b) that $g \simeq g_l$. This regime corresponds to the paths that pass through points far away from the region of producers. In all studied cases (injector symmetric or asymmetric, injector outside) the long paths are responsible for a power-law behavior with exponent $g \simeq g_l$.

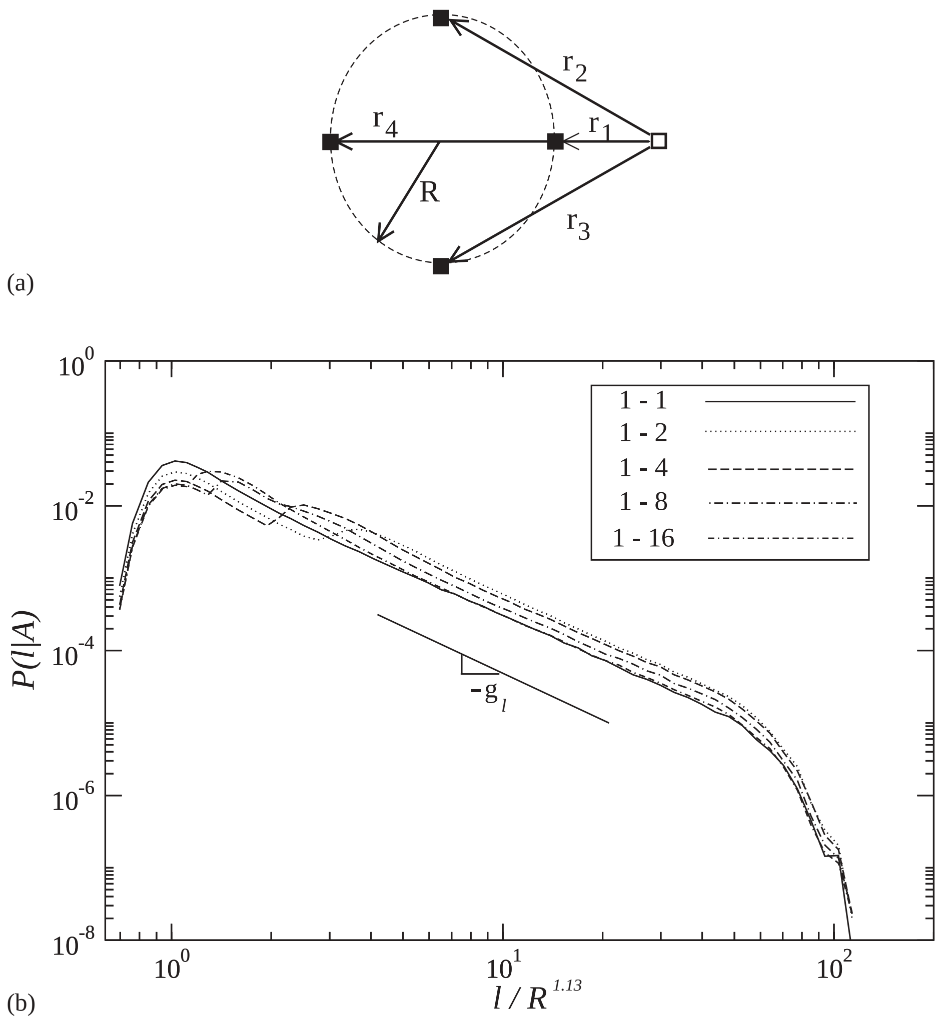


Fig. 4. The case of an injector outside an array of producers. (a) The layout of the array of wells: the injector (empty square) is external to a circle of 4 producers (filled squares). The producers are over a circle (dashed line) of radius R. The Euclidean distances r_i, $1 \leqslant i \leqslant 4$ are indicated by arrows. (b) Log–log plot of shortest path distribution, $P(l|A)$, versus $l/R^{d_{\min}}$ at the percolation threshold. In the simulations $L = 500$, $R = 16$ and $R_1 = 16$. Five cases corresponding to 1, 2, 4, 8, and 16 production wells are indicated in the figure.

4. Final remarks

In this work we study the probability distribution of shortest paths in problems of multiple oil wells using percolation techniques. This problem is an extension of the distribution, $P(l|r)$, of shortest chemical path l on the percolating cluster for the case of one injection well and one production well separated by an Euclidean distance r. This analysis is an important problem in oil industry since the typical oil field is more complex than a simple pair injector-producer. We perform extensive simulations for one injector and different arrays of producers and estimate the distribution $P(l|A)$.

A symmetric array of producers around a single injector leads to a single maximum in $P(l|A)$ while the asymmetric cases lead to multiple maxima. We observe that the number of maxima is related to the number of distinct distances from the injector to a set of producers. In this asymmetric case some of the maxima may overlap. This phenomenon is seen essentially for the second maximum and the next ones. It can be understood because the partial peaks in the distributions become broader for producers located far away, while the separation between maxima is reduced with the increasing number of producers.

In the case of multiple producers, we observe two regimes after the first peak separated by a crossover distance l_c. The first regime, $r_1^{d_{\min}} < l < l_c$, for the asymmetric case, was described in the previous paragraph. The first regime for the symmetric case is characterized by a fast decay in $P(l|A)$. In this regime the equidistant producers compete to be the endpoints of the paths. Short paths are more probable for multiple producers than in the standard case of one producer. When there are multiple producers the global maximum of $P(l|A)$ is higher and occurs for lower l/R compared with the single producer case. Also the decay of $P(l|A)$ after the maximum will be faster. The second regime, $l_c < l < L^{d_{\min}}$, is the same for symmetric and asymmetric cases. This regime is characterized by rare long paths that escape from the region of producers but eventually return and reach a producer. In fact these long paths see the set of producers as a mean field single producer and as a consequence the exponent of the power-law is $g \simeq g_l$.

Our findings are relevant in analyzing the consequences of well geometry to oil production. In future papers we intend to treat geometries other than those treated here using percolation theory. Also we plan to introduce other stochastic models [9] in order to tackle more realistic problems.

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