# Critical Indices for a System of Spins of Arbitrary Dimensionality Situated on a Lattice of Arbitrary Dimensionality

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A model Hamiltonian $\mathcal{H}=-J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j$ is considered, where $\mathbf{S}_i$ are isotropically interacting $D$-dimensional classical spins. Thus the $S=\frac{1}{2}$ Ising, classical planar, and classical Heisenberg models are obtained if $D=1$, 2, and 3, respectively; moreover, as $D\rightarrow\infty$, the free energy corresponding to $\mathcal{H}$ approaches the free energy of the exactly-soluble spherical model of Berlin and Kac. $\mathcal{H}$ can also be solved exactly providing the lattice dimensionality $d=1$. For lattices of higher dimensionality, we must resort to approximation techniques; hence we calculate high-temperature expansions for arbitrary $D$ and $d$ of the susceptibility, internal energy (specific heat), and the second moment. (With a few exceptions, such expansions have been available in the past only for $D=1$, 3 and $d=1$–3.) Somewhat smoother series to facilitate more reliable extrapolation are then obtained by re-expanding all of the series in terms of the nearest-neighbor spin correlation function for a one-dimensional lattice. We then study the dependence on spin and lattice dimensionality of the critical temperatures $T_c(D,d)$ and the critical exponents $\gamma(D,d)$ [$(\chi\sim(T-T_c)^{-\gamma}$], $\alpha(D,d)$ [$C\sim(T-T_c)^{-\alpha}$], $\nu(D,d)$ [inverse correlation range $\kappa\sim(T-T_c)^{-\nu}$]. These "critical properties" are all found to vary monotonically (and in most cases, smoothly) with $D$ and $d$. The assumptions behind making critical phenomena predictions from a dozen or so terms (of an infinite series!) are given increased plausibility by the fact that we have extended the various expansions for the spherical model to 80–100 terms and find that none of the predictions based upon $\sim$10 terms is changed [e.g., $T_c(\infty,2)=0$, $\gamma(\infty,3)=2$, etc.].

It is by now fairly widely believed that critical exponents depend *primarily* on lattice dimensionality $d$ and, of course, on the interaction Hamiltonian. For example, the variation of $\gamma$ with spin quantum number $S$, if real, is barely outside the "experimental error" inherent in the extrapolation procedures. Consider, then, the Hamiltonian[1]

$$\mathcal{H}=-J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j, \tag{1}$$

where $\mathbf{S}_i$ are isotropically-interacting $D$-dimensional classical spins of magnitude $D^{1/2}$. Clearly (1) reduces for $D=1$, 2, and 3 to the $S=\frac{1}{2}$ Ising, classical planar, and classical Heisenberg models. The properties of (1) are known exactly in 3 different limits: (i) as $D\rightarrow\infty$ (1) reduces to the exactly-soluble spherical model,[2] (ii) as $d\rightarrow\infty$, one obtains "essentially" mean field results,[3] and (iii) if $d=1$, the system (1) can be solved exactly for all $D$.[4]

One feature of the model Hamiltonian (1) is that a single diagrammatic representation suffices to calculate the various high-temperature expansions for all values of $D$ and $d$. Hence we have obtained general-lattice expressions for arbitrary $D$ for the coefficients in the high-T expansions of the susceptibility $\chi$, internal energy (specific heat $C$), and second moment.[5]

Generally, we found that slightly smoother series were obtained if we re-expanded each series in terms of the parameter $y_D$, the n.n. spin correlation function for a system of $D$-dimensional spins situated on a one-dimensional lattice. For example, $y_\infty\equiv x/[(1+x^2)^{1/2}+1]$ where $x\equiv 2J/kT$; the first few coefficients in the new series for $D=\infty$ are given for some $d$-dimensional lattices in Table I.

TABLE I. The first few coefficients in the new susceptibility expansion (in terms of $y_D$) for $D=\infty$, obtained from the general-lattice, arbitrary-$D$ expressions. The series for $D=\infty$ have recently been extended to order $n=100$ [H. E. Stanley (unpublished)].

| $n$ | $d=2$ | $d=3$ | $d=4$ | $d=5$ |
|---|---|---|---|---|
| 1 | 4 | 6 | 8 | 10 |
| 2 | 12 | 30 | 56 | 90 |
| 3 | 36 | 150 | 392 | 810 |
| 4 | 100 | 726 | 2 696 | 7 210 |
| 5 | 260 | 3 462 | 18 440 | 64 010 |
| 6 | 644 | 16 230 | 125 192 | 566 090 |
| 7 | 1 508 | 75 318 | 847 496 | 5 000 810 |
| 8 | 3 380 | 345 990 | 5 714 024 | 44 094 410 |
| 9 | 7 236 | 1 577 958 | 38 456 072 | 388 568 010 |
| 10 | 14 828 | 7 147 806 | 258 171 128 | 3 420 464 090 |

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[1] Hamiltonian (1) was first considered (to the best of our knowledge) in H. E. Stanley, Phys. Rev. Letters **20**, 589 (1968).

[2] H. E. Stanley, Phys. Rev. **176**, 718 (1968).

[3] This is at present a plausible but unproved hypothesis. See, e.g., the $D=1$ work of M. E. Fisher and D. S. Gaunt, Phys. Rev. **133**, A224 (1964).

[4] H. E. Stanley, Phys. Rev. **178**, 570 (1969); Proc. of the International Conf. on Statistical Mechanics, J. Phys. Soc. Japan (in press).

[5] These general-$D$, general-lattice expressions are too lengthy to reproduce here; they will be presented elsewhere.

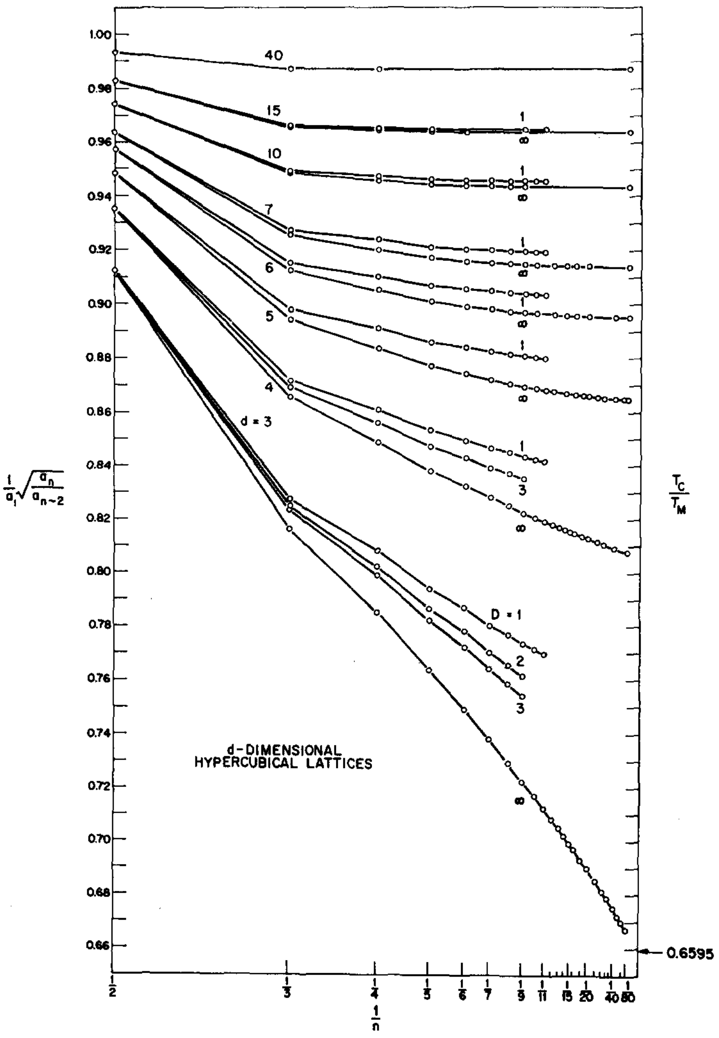


FIG. 1. Ratio plot for the susceptibility series indicating the variation of $T_c(D, d)$ with $D$ and $d$. The additional terms for the spherical model $(D=\infty)$ come from a direct expansion of the analytic expressions themselves [H. E. Stanley (unpublished)], and the additional terms for $D=1$ come from Ref. 3.

TABLE II. Estimates of critical properties for $D$-dimensional spins situated on a three-dimensional (fcc) lattice. The variation with $D$ was found to be monotonic (and smooth) for $D$ between 1 and $\infty$. The entries in the sixth and seventh columns are predicted to be zero by various of the scaling relations; that the departure from zero appears to be a monotonically decreasing function of spin dimensionality should be viewed with caution—the errors in these small numbers are roughly as large as the numbers themselves. The quantity "$\beta$" tabulated in the eighth column is obtained by assuming the validity of the scaling relation $\alpha+2\beta+\gamma=2$. The fact that "$\beta$" so defined turns out to be $\gamma/4$ (regardless of whether or not we choose $\alpha$ and $\gamma$ to be the "closest rational fraction") suggests the validity of the scaling relation $\gamma=\beta(\delta-1)$ with $\delta=5$. If we assume the validity of $\gamma=\beta(\delta-1)$ [or, equivalently, of $\alpha+\beta(\delta+1)=2$], we obtain the values of "$\delta$" given in the last column.

| $D$ | $T_c/T_M$ | $\gamma$ | $\alpha$ | $\nu$ | $\eta\equiv2-\gamma/\nu$ | $d\nu-(2-\alpha)$ | $\beta\equiv(2-\alpha-\gamma)/2$ | $\delta\equiv\frac{2-\alpha+\gamma}{2-\alpha-\gamma}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.816 | 1.25 | 0.125 | 0.643 | 0.056 | 0.054 | 0.3125(5/16) | 5 |
| 2 | 0.804 | 1.32(~21/16) | 0.02(~1/32) | 0.675 | 0.04 | 0.04 | 0.33(~21/64) | 5 |
| 3 | 0.793 | 1.38(~11/8) | −0.07(~−1/16) | 0.70 | 0.03 | 0.03 | 0.345(~11/32) | 5 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $\infty$ | 0.7436 | 2 | −1 | 1 | 0 | 0 | $\frac{1}{2}$ | 5 |

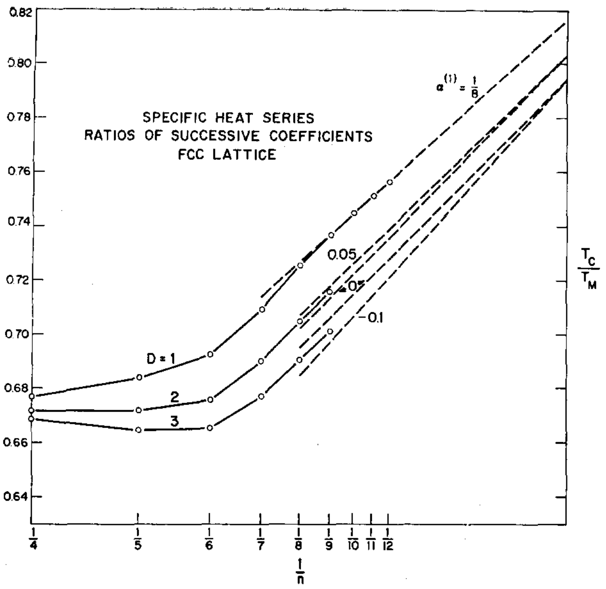


FIG. 2. Ratios of successive coefficients in the specific-heat series for $d=3$, and $D=1$, 2, and 3. If $C\sim(T-T_c)^{-\alpha}$, then the limiting slope should be proportional to $\alpha-1$. The dashed lines indicate the limiting asymptotes which would be obtained for various values of $\alpha$, assuming that $T_c$ (the $n=\infty$ intercept) is given by the (much more reliable) susceptibility series. The additional terms for $D=1$ were obtained by M. F. Sykes, J. L. Martin, and D. L. Hunter, Proc. Phys. Soc. (London) 91, 671 (1967).

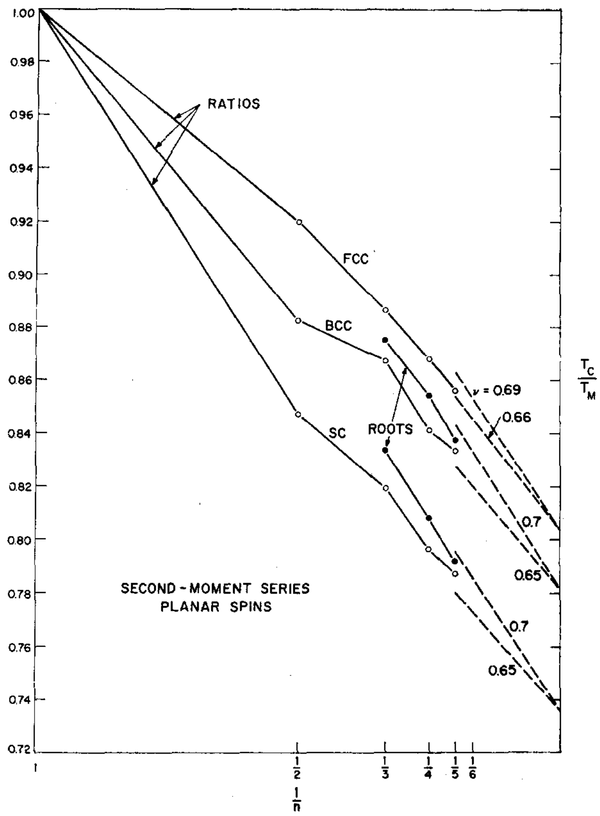


FIG. 3. Ratios of successive coefficients and square roots of ratios of alternate coefficients in the second moment series (the shortest of all the series calculated). Our estimate $\nu(2, 3)=0.675\pm0.025$ agrees with that of D. Jasnow and M. Wortis (unpublished) for the classical $x$–$y$ model.

Next, numerical estimates for all spin dimensionalities $D$ and all lattice dimensionalities $d$ of $T_c(D, d)$, $\gamma(D, d)$, $\alpha(D, d)$, and $\nu(D, d)$ were obtained by utilizing standard extrapolation procedures (see, e.g., Figs. 1–3).[6] The variation of these properties with $D$ for fixed $d(d=3)$ is illustrated in Table II. The various critical properties appear to vary monotonically (and, at least for $d=3$, smoothly) with $D$.[7] One consequence of this apparent "monotonicity of critical indices" is that the spherical model, which in the past has been interpreted as a soluble approximation to the Ising model, would appear to be a much better approximation to the more "realistic" Heisenberg model.

[6] In particular, we note from Fig. 1 that as *d increases*, the difference between the critical temperatures for different values of *D decreases* (as does the difference between the susceptibility exponents, which are determined by the limiting slopes). Thus the larger the lattice dimensionality, the better the spherical model approximates the behavior of, say, three-dimensional spins.

[7] That $T_c(D, 3)$ should decrease with increasing $D$ is at least intuitively clear.