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Scale-invariant properties of public-debt growth

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Abstract – Public debt is one of the important economic variables that quantitatively describes a nation's economy. Because bankruptcy is a risk faced even by institutions as large as governments (*e.g.*, Iceland), national debt should be strictly controlled with respect to national wealth. Also, the problem of eliminating extreme poverty in the world is closely connected to the study of extremely poor debtor nations. We analyze the time evolution of national public debt and find "convergence": initially less-indebted countries increase their debt more quickly than initially more-indebted countries. We also analyze the public debt-to-GDP ratio $\mathcal{R}$, a proxy for default risk, and approximate the probability density function $P(\mathcal{R})$ with a Gamma distribution, which can be used to establish thresholds for sustainable debt. We also observe "convergence" in $\mathcal{R}$: countries with initially small $\mathcal{R}$ increase their $\mathcal{R}$ more quickly than countries with initially large $\mathcal{R}$. The scaling relationships for debt and $\mathcal{R}$ have practical applications, *e.g.* the Maastricht Treaty requires members of the European Monetary Union to maintain $\mathcal{R} < 0.6$.

Introduction. – Just as an individual is expected to control his/her debt to asset ratio, so is a government expected to control its national debt as a function of the country's wealth, measured, *e.g.*, by its gross domestic product (GDP). In a dynamic global economy, excessive borrowing cannot persist indefinitely, as creditors are bound to call in large loans. While a country suffers financial problems when its GDP does not increase fast enough, even more serious trouble begins when its debt increases faster than its GDP. While national GDP has been the topic of many studies on economic growth [1–3], the empirical analysis of public debt has lagged due to lack of comprehensive data.

Large sets of public-debt data, dating back several decades, and ranging from poor to rich countries, have recently become available. Here we use concepts of statistical physics to analyze public-debt data for a wide cross-section of economies including underdeveloped, developing, and developed countries. The total public-debt data, along with total GDP data, are available at the *Inter-American Development Bank* [4], and are compiled and analyzed in refs. [5,6]. Population data are available

by the *World Bank*, and can be reconstructed through the GDP and *per capita* GDP data compiled in ref. [7]. We deflate all USD amounts to units of the USD in the year 2000. In our analysis, we compare public debt only within the same country, in order to avoid any differences in the theoretical and practical definition of debt and the reporting of debt by various countries, an issue pointed out in refs. [5,8]. Hence, our results are robust with respect to mis-reporting and ambiguous definitions of public debt [8]. In fig. 1 we plot the debt-to-GDP ratio $\mathcal{R}$ for many countries, grouped by common historical, geographical, and financial factors. In fig. 1(f) we plot the average $\mathcal{R}$ calculated for three subgroups, according to *World Bank* Income Group (IG) classifications, and observe relatively high levels of $\mathcal{R}$ among the poorest countries.

Economic-growth theories predict that GDP should "converge" towards equality, with wealthy countries experiencing smaller relative growth rates than poor countries. However, the opposite has been found for economic-wealth data [9–11]. So we address the question, what are the growth dynamics for public debt? To answer this question, we analyze a comprehensive database of national public debt and GDP to investigate the dynamics of debt growth and growth in $\mathcal{R}$.
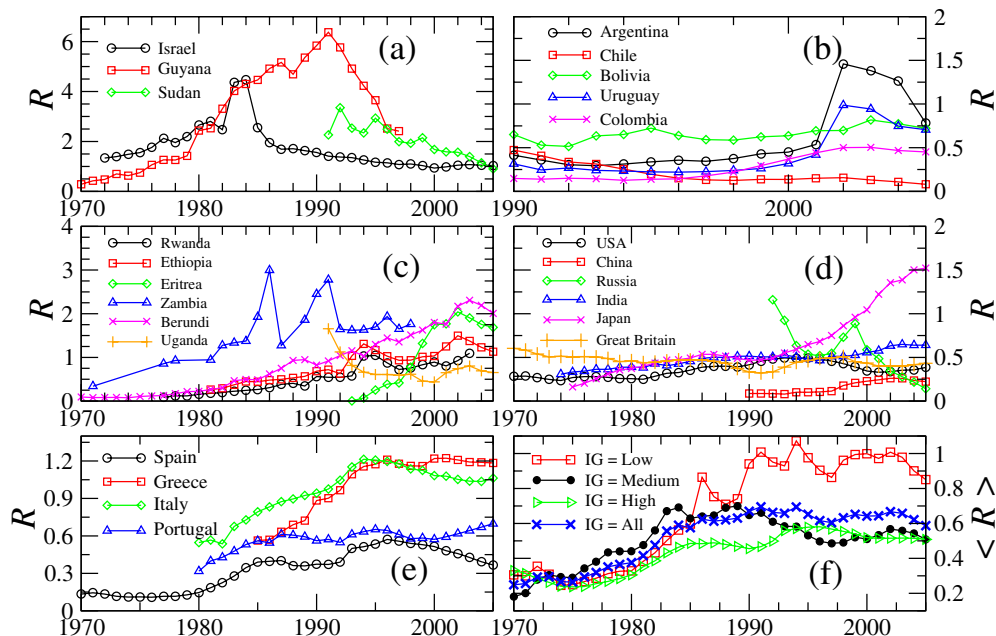
Fig. 1: (Colour on-line) Illustration of debt-to-GDP ratio $\mathcal{R}(t)$ for (a)–(e) several countries and (f) the average trend for three subsets according to World Bank Income Group (IG) classifications. (a) Countries with turmoil, as in the case of Israel and Sudan which experienced periods of war, and Guyana which formed a new government, can experience periods of extremely high $\mathcal{R}$. (b) The Argentine financial crisis of the late 1990s placed many South American countries in a state of debt stress. (c) The debt burden in many African countries, some which are also plagued by war and political turmoil, is increasing with time. Many of these countries are candidates for foreign debt cancellation (jubilee). (d) Six large countries. (e) Mediterranian countries with recent concern of default risk. (f) The average $\mathcal{R}$ is increasing with time, with low-income countries currently at relatively high levels reflecting the burden of unsustainable debt.

With the current global credit crunch, and several notable recent national defaults, it is important to address sustainable public debt, defined as the amount of debt where the receiving country is capable of meeting its current and future debt obligations [12–14]. The total current government debt $D(t)$ increases from last year's debt $D(t-1)$ partially due to interest payments on the debt $D(t-1)$ at interest rate $I_D(t)$, and partially because of the current primary deficit, defined as the difference between spending $S(t+1)$ and taxes $T(t+1)$. Thus

$$D(t) = [1 + I_D(t-1)]D(t-1) + [S(t) - T(t)]. \quad (1)$$

We consider three possible scenarios for public-debt growth dynamics:

i) Growth rates of the country debt do not depend on the initial debt level.

ii) A more-indebted country has a *larger* debt growth rate than a less-indebted country, so that relative differences between debt across countries increases over time (*divergence*).

iii) A more-indebted country has a *smaller* debt growth rate than a less-indebted country, so that relative differences between debt across countries decreases over time (*convergence*).

These three scenarios have different implications for investors, who will only accept government debt up to some ceiling. Hence, one would expect that more-indebted countries would increase their debt more slowly than less-indebted countries.

Empirical results. – To ascertain which of the three debt scenarios is better supported by empirical facts, we define for country i the annualized logarithmic growth rate of *per capita* initial debt $d_i(t)$ between years t and $t + \Delta t$

$$r_i(t, t + \Delta t) \equiv \frac{\log[d_i(t + \Delta t)/d_i(t)]}{\Delta t} . \quad (2)$$

We compare $r_i(t, t + \Delta t)$ to $d_i(t)$, assuming $r_i(t, t + \Delta t)$ depends on debt size by

$$r_i(t, t + \Delta t) \cong \alpha - \beta \log[d_i(t)]. \quad (3)$$

The functional form of eq. (3) can also be expressed as

$$\log[d_i(t + \Delta t)] = \alpha \Delta t + (1 - \beta \Delta t) \log[d_i(t)]. \quad (4)$$

If $\beta > 0$, there is convergence in *per capita* debt data across countries, since initially more-indebted economies tend to increase their debt slower (smaller $r_i(t, t + \Delta t)$) than initially less-indebted economies. Hence, β represents the "speed of convergence", a concept introduced for
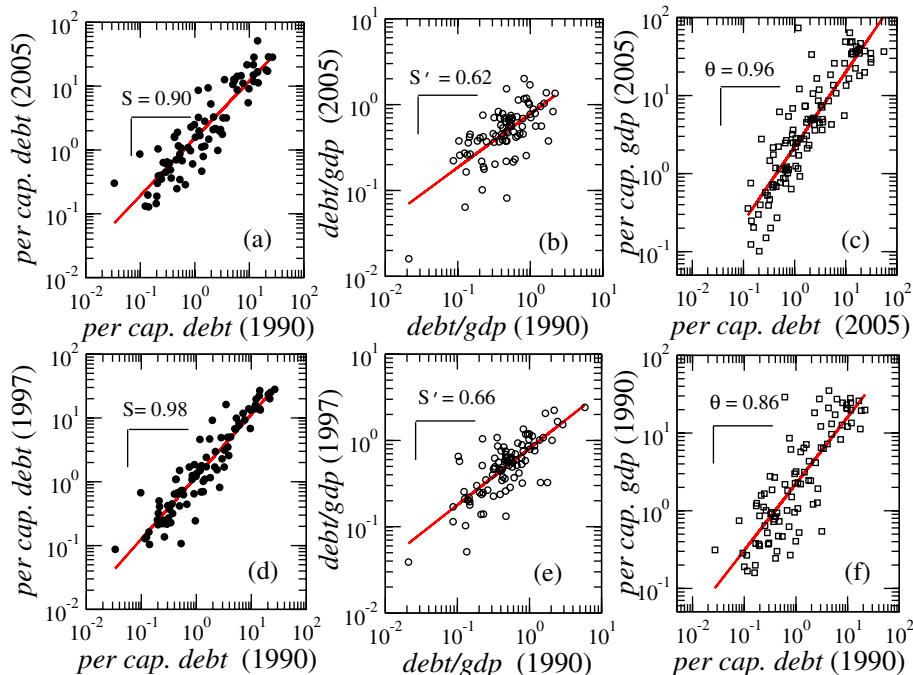
Fig. 2: (Colour on-line) The 3 possible relations between *per capita* debt, *per capita* GDP, and $\mathcal{R}$ for a set of low-income, medium-income, and high-income countries. Regression of *per capita* debt $d(t)$ in 1990 *vs.* *per capita* debt $d(t + \Delta t)$ for (a) $\Delta t = 15$ years and (d) $\Delta t = 7$ years later indicates that there is *convergence* in the *per capita* debt data across the countries over the larger time horizon in panel (a) with $S \equiv 1 - \beta \Delta t < 1$ implying $\beta > 0$. Regression of $\mathcal{R}(t)$ in 1990 *vs.* $\mathcal{R}(t + \Delta t)$ for (b) $\Delta t = 15$ years and (e) $\Delta t = 7$ years indicates that there is also *convergence* over these time horizons. (c), (f) We also illustrate the scale-invariant relation between between *per capita* debt and *per capita* GDP in units of 10^3 USD per person in the year 2000.

per capita GDP data in ref. [11]. A larger positive value of β results in faster convergence, equalizing the *per capita* debt across all countries more quickly. If $\beta < 0$ there is divergence in debt data, where initially less-indebted countries with smaller $d_i(t)$ increase their debt slower than initially more-indebted countries.

In fig. 2 we plot the 1990 *per capita* debt of more than 80 countries, representing low-, medium-, and high-wealth countries, considering several relationships. First, we compare the *per capita* debt over (a) 15-year and (d) 7-year time horizons. We find the slope $S = (1 - \beta \Delta t)$ of the regression in eq. (4) is less than one, requiring $\beta > 0$ which corresponds to scenario iii).

To confirm the convergence across countries for other time horizons, fig. 3 shows the value of S for varying initial $d(t)$ and time horizon Δt in eq. (4). We find $S \equiv 1 - \beta \Delta t < 1$ for most horizons Δt, implying convergence, where less-indebted countries increase their debt faster than more-indebted countries. However, there is a period in the beginning of the 1990s that is the exception, with $S > 1$ and $\beta < 0$. This period of divergence in *per capita* debt may be related to the 20-year lows in interest rates which may have resulted in increased borrowing, even among heavily indebted countries. Since 1995, the values of S have returned to values less than one, indicating a return to convergence.

A natural question is: how does the *per capita* debt d_i vary across all countries and by income group? Power law probability density functions (pdf) have been observed for total country GDP [15] and *per capita* GDP [16]. Figure 4 shows the pdf $P(d)$ for all countries analyzed over the 36-year period 1970–2005. We observe large variations across income groups, where low-income countries typically have relatively small *per capita* debt values reflecting their small *per capita* borrowing capacity. In contrast to the Zipf rank-GDP curves with $\zeta_{GDP} \approx 1$ corresponding to a pdf scaling exponent $1 + 1/\zeta_{GDP} \approx 2$ [15], we observe in fig. 4 (inset) a scaling value $\zeta_d \approx 0.3$ corresponding to a relatively large pdf scaling exponent $1 + 1/\zeta_d \approx 4.3$.

In a country where both GDP and debt grow with time, one must analyze the dynamics of both debt and GDP. Since a debt that is large for Luxembourg is not large for the US, various indices have been proposed in order to compare the burden of debt to the ability of the country's economy to generate income. These include $\mathcal{R}$ [17], so we apply the convergence analysis of eq. (4) to $\mathcal{R}(t)$ obtaining

$$\log[\mathcal{R}_i(t + \Delta t)] = \alpha' \Delta t + (1 - \beta' \Delta t) \log[\mathcal{R}_i(t)]. \quad (5)$$

Figure 2 compares $\mathcal{R}(t)$ over (b) 15-year and (e) 7-year time horizons. Figure 5 shows $S' \equiv (1 - \beta' \Delta t) < 1$, implying convergence $\beta' > 0$, over a large range of Δt-year horizons for initial year t.
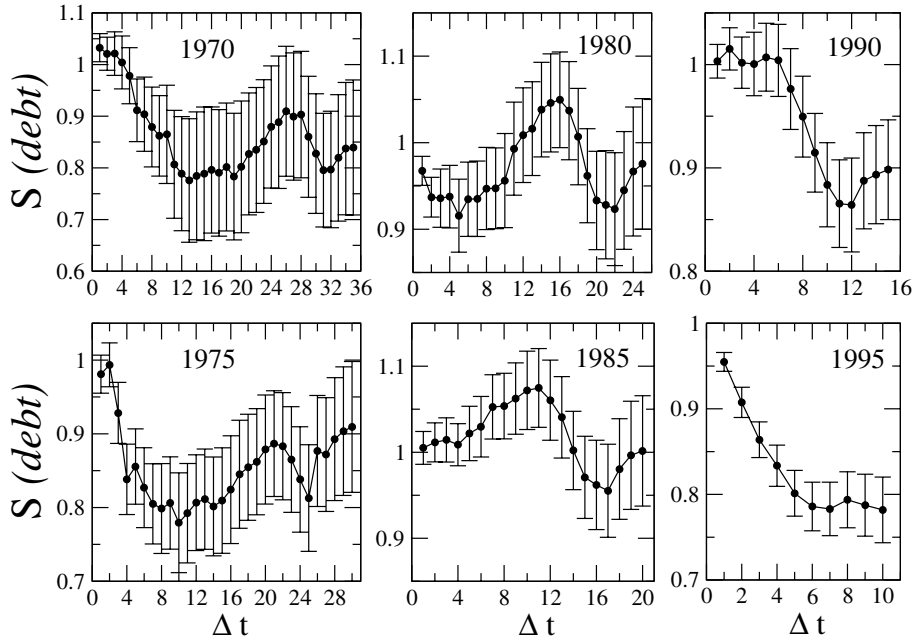
Fig. 3: The slope $S \equiv 1 - \beta\Delta t$ of the log-log regression in eq. (4) is typically less than one for a given time horizon of Δt years after the initial year t labeling each panel. Values of $S < 1$ imply *convergence* ($\beta > 0$) in *per capita* debt over the Δt year period. R^2 values for log-log regressions are typically > 0.7 for $10 < \Delta t < 20$ years, and are typically > 0.8 for $1 < \Delta t < 10$ years, while the number of countries for each regression varies in the ranges 22–25 (1970) and 103–120 (1995). Interestingly, we observe a short period of *divergence*, where for initial years 1980 and 1985, the value of $S > 1$ for $\Delta t \approx 12$ years corresponding to the mid-1990s.
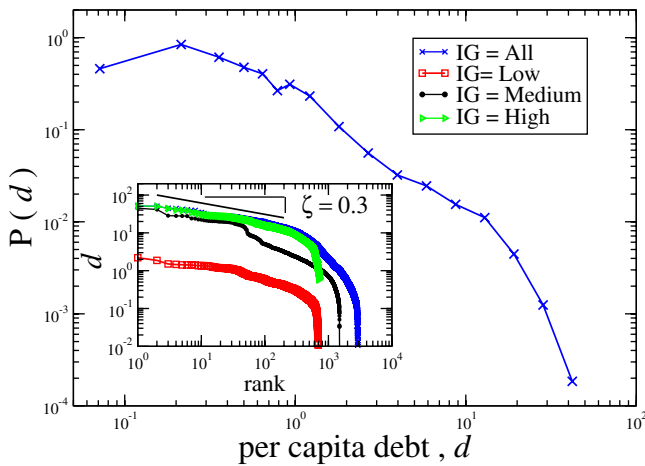

Fig. 4: (Colour on-line) Right-skewed probability density function $P(d)$ of *per capita* debt d for all countries analyzed over the 36-year period 1970–2005 in units of 10^3 USD per person in the year 2000. Inset: Zipf rank-frequency plot for d values for all countries and for three subgroups according to *World Bank* Income Group (IG) classification. Countries in the high and medium IG categories contribute to the wide range of d values, whereas poor countries have relatively small d values reflecting their small *per capita* borrowing capacity. We also show a power law with exponent $\zeta = 0.3$ for comparison.

A responsible government is expected to monitor simultaneously the growth of debt and GDP [18]. By borrowing money, a country may increase $\mathcal{R}(t)$ for some time,

but clearly $\mathcal{R}(t)$ cannot increase indefinitely, as increased debt can negatively affect GDP growth [19]. $\mathcal{R}(t)$ is an important quantity for determining the ability of a debtor to make debt payments. For large $\mathcal{R}(t)$ there is a larger probability that either the debtor will not be able to make timely payments or be able to prevent further debt increase with time, two scenarios that lead to credit default. Debtor default risk is estimated by many rating agencies and financial organizations. In order to quantify the risky debt levels, we collect the $\mathcal{R}(t)$ values of all countries analyzed over the 36-year period 1970–2005 and plot the pdf $P(\mathcal{R})$ in fig. 6. We find $\langle \mathcal{R} \rangle = 0.57 \pm 0.54$, and we fit the pdf to a Gamma distribution $P(\mathcal{R}) \propto \mathcal{R}^{k-1} \exp[-\mathcal{R}/\mathcal{R}_c]$ with $k = 2.0 \pm 0.1$ and $\mathcal{R}_c = 0.30 \pm 0.01$, using the maximum-likelihood estimator. The extreme value statistics of the Gamma pdf can be used to define thresholds for sustainable debt.

In order to analyze the countries that have large $\mathcal{R}(t)$ and a high risk of default, the countries which constitute the pdf tail, we plot rank-frequency curves in fig. 6 (inset). The Zipf plots show a power law over three orders of magnitude, with scaling exponent $\zeta_{\mathcal{R}} \approx 0.4$ corresponding to $P(\mathcal{R}) \sim \mathcal{R}^{-3.5}$ in the tail regime.

Model. – Our analysis, performed across a wide cross-section of countries, confirms the existence of convergence in public debt. This is opposite of what is in GDP data [9–11], where the speed of convergence β is negative. We now discuss how to model the scaling result we obtain, and how to use the scaling result obtained for GDP and
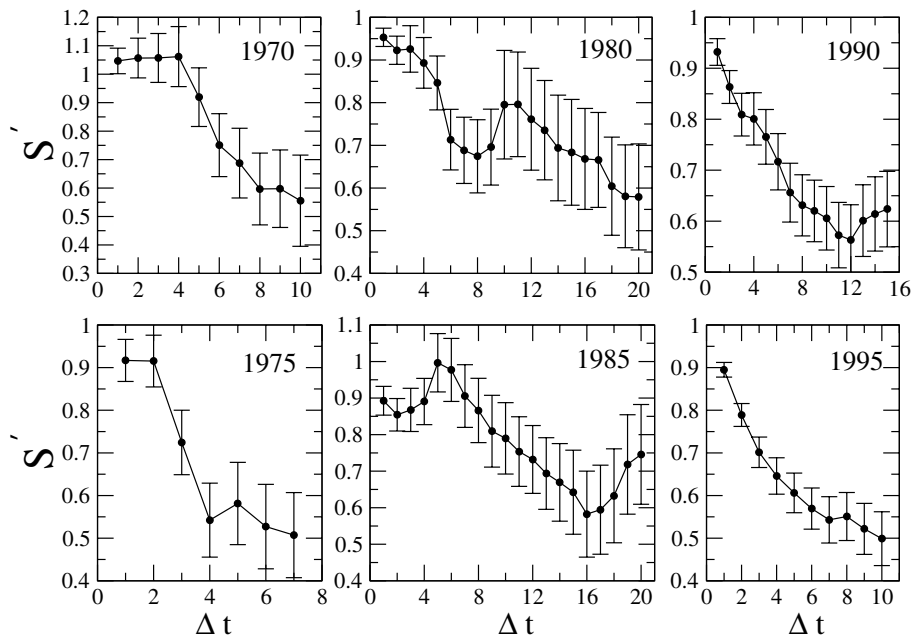
Fig. 5: The slope $S' \equiv 1 - \beta' \Delta t$ of the log-log regression in eq. (5) of $\mathcal{R}$. We find $S' < 1$ a given time horizon of Δt years after the initial year t labeling each panel. Values of $S < 1$ imply *convergence* ($\beta > 0$) in *per capita* debt over the Δt year period. We plot values of S' only for log-log regressions with $R^2 > 0.3$. The number of countries for each regression varies in the ranges 22–25 (1970) and 103–120 (1995).
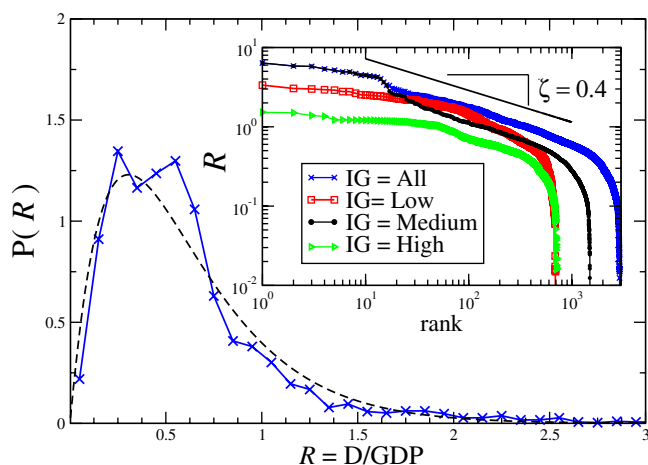

Fig. 6: (Colour on-line) Probability density function $P(\mathcal{R})$ for all countries analyzed over the 36-year period 1970–2005. For comparison, we approximate $P(\mathcal{R})$ with a Gamma distribution with parameters $k = 2.0$ and $\mathcal{R}_c = 0.30$. The extreme statistics of $P(\mathcal{R})$ can quantify the threshold of sustainable debt. Inset: Zipf rank-frequency plot for $\mathcal{R}$ values. We separate data into three subgroups by *World Bank* Income Group (IG) classification to demonstrate the range of large $\mathcal{R}$ values within different wealth groups. The straight line is a power law with exponent $\zeta = 0.4$ for comparison.
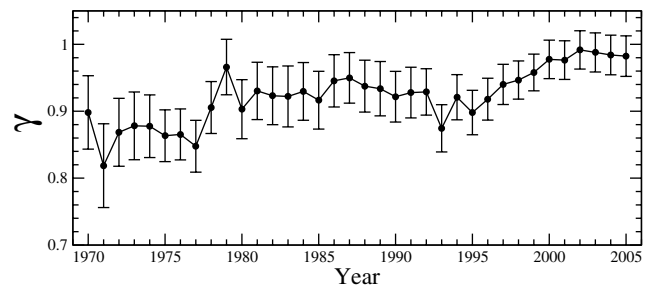

Fig. 7: Annual trend of power law exponent γ, which quantifies the scaling relationship between debt and GDP as $g \sim Ad^\gamma$ in eq. (6). We find $\gamma < 1$ which reflects the burden of debt on country GDP.

public debt. Figures 2(c) and (f) compare the *per capita* debt to *per capita* GDP for the years 1990 and 2005. The typical relationship between debt and GDP shows a scale-invariant form,

$$g \sim Ad^\gamma, \qquad (6)$$

where g is the *per capita* GDP and d is the *per capita* debt. In fig. 7 we plot the values of γ for the set of countries analyzed in each yearly data set. We find $\gamma \lesssim 1$ which reflects the burden of debt on the country GDP.

We may consider the dynamics of public debt by assuming that the government borrows $B(t)$, a fixed proportion of GDP given by $B(t) \equiv D(t) - D(t-1) \equiv \Delta D(t) = bG(t)$, with constant deficit ratio b [20]. Then $r_D \equiv \Delta D/D = bG(t)/D(t)$, and

$$r_d = bg(t)/d(t) - r_{pop} = bN/d(t)^{1-\gamma} - r_{pop}, \qquad (7)$$

where r_{pop} denotes the population growth rate [17]. Hence,

$$\frac{\partial r_d}{\partial d} = \frac{-bN(1-\gamma)}{d^{2-\gamma}} \propto 1/d^{2-\gamma}. \qquad (8)$$

We observe from fig. 7 that $\gamma \lesssim 1$ so that $\partial r_d / \partial d \simeq -1/d$. For this reason, we use the regression $r_d \cong \alpha - \beta \log[d_i(t)]$, which agrees well with the data in figs. 2 and 3.

Discussion. – We demonstrate *convergence* in *per capita* public debt across a wide set of countries during the period 1970–2005, a result of general interest for economists, complex-systems researchers, and creditors. Some experts believe that convergence across all countries is possible through globalization [15,21–25] and access to open markets [26]. While public debt can be used to invest in a country's development via physical infrastructure, technology, and social programs, its use requires responsible governance. Corruption [27,28] and the misuse of public debt can lead to insurmountable debt contributing to financial crisis, which can cause further increase in debt levels through exchange rate depreciation [29,30]. There are also instances of extremely poor debtor nations that are unable to meet their current and future debt obligations (see fig. 1). Recent programs such as the Heavily Indebted Poor Countries (HIPC) Initiative, sponsored by the World Bank and the IMF, and the Jubilee 2000 Campaign-to-Drop-the-Dept, have called on debt cancellation for extremely poor debtor nations as a crucial step in the UN Millennium Project to eliminate extreme poverty [31–33]. Further, debt has become a problem for not only the extremely poor countries. With the current global credit crunch, their is an increased need for responsible use of government debt.

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