Fractionally integrated process with power-law correlations in variables and magnitudes

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Motivated by the fact that many empirical time series—including changes of heartbeat intervals, physical activity levels, intertrade times in finance, and river flux values—exhibit power-law anticorrelations in the variables and power-law correlations in their magnitudes, we propose a simple stochastic process that can account for both types of correlations. The process depends on only two parameters, where one controls the correlations in the variables and the other controls the correlations in their magnitudes. We apply the process to time series of heartbeat interval changes and air temperature changes and find that the statistical properties of the modeled time series are in agreement with those observed in the data.

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I. INTRODUCTION

Many physical, physiological, biological, and social systems are characterized by complex interactions among many different individual components leading to scale-invariant correlations [1–4]. Since the resulting observable variable in the output of these systems at each moment is the product of its magnitude and sign, recent investigations have focused on the study of correlations in magnitude and sign time series [5–12]. For example, time series of changes $\delta\tau_i$ of heartbeat intervals [6–8], river flux values [9], physical activity levels [10], and intertrade times in the stock market [11] exhibit power-law anticorrelations, while their magnitudes $|\delta\tau_i|$ are positively correlated. A second correlation pattern is observed for human gait dynamics, where increments of interstride intervals are power-law anticorrelated, while their magnitudes are uncorrelated [12]. A third correlation pattern is observed for financial data, where price changes are uncorrelated, while their magnitudes are long-range power-law correlated [5].

Several stochastic processes, including the autoregressive fractionally integrated moving average (ARFIMA) process [13,14] and the fractionally integrated autoregressive conditional heteroscedastic (FIARCH) process [15], have been proposed to generate time series with power-law correlations in $\delta\tau_i$ or power-law correlations in $|\delta\tau_i|$. The ARFIMA process was introduced to generate time series with power-law correlations or anticorrelations in $\delta\tau_i$. We will show in Sec. III that the ARFIMA process does not exhibit magnitude correlations if the variables are anticorrelated. Hence the ARFIMA process is an appropriate candidate for modeling human gait dynamics. The FIARCH process was introduced in finance to model uncorrelated time series with power-law magnitude correlations. However, neither the ARFIMA process nor the FIARCH processes is capable of modeling time series that are simultaneously power-law anticorrelated in $\delta\tau_i$ and power-law correlated in $|\delta\tau_i|$, a case found in many time series such as changes of heartbeat intervals, physical activity levels, intertrade times in finance, or river flux values.

II. DEFINITIONS AND METHODS

With the goal of constructing a stochastic process that can simulate time series $\delta\tau_i$ with power-law anticorrelations in $\delta\tau_i$ and power-law correlations in $|\delta\tau_i|$, we combine the ARFIMA process with the FIARCH process and define process $\mathcal{A}(\rho_1,\rho_2)$ by

$$\delta\tau_i = \sum_{n=1}^{\infty} a_n(\rho_1)\,\delta\tau_{i-n} + \sigma_i\eta_i, \tag{1a}$$

$$\sigma_i = \sum_{n=1}^{\infty} a_n(\rho_2)\frac{|\delta\tau_{i-n}|}{\langle|\delta\tau_i|\rangle}, \tag{1b}$$

$$a_n(\rho) = \rho\frac{\Gamma(n-\rho)}{\Gamma(1-\rho)\Gamma(1+n)}. \tag{1c}$$

Here $\rho_1 \in (-0.5, 0.5)$ and $\rho_2 \in [0, 0.5)$ are free parameters, Γ denotes the Gamma function, and η_i denotes independently and identically distributed Gaussian variables with expectation value $\langle\eta_i\rangle=0$ and variance $\langle\eta_i^2\rangle=1$. For $\rho\in(0,0.5)$, the weights $a_n(\rho)$ satisfy the constraint $\Sigma_{n=1}^{\infty}a_n(\rho)=1$, and by using the Stirling formula it can be shown that the weights scale as $a_n(\rho)\propto n^{1-\rho}$ for asymptotically large values of n.

To eliminate trends in empirical data, one commonly takes first-order differences x_i-x_{i-1}, or calculates higher-order integer differences. This differencing procedure is accomplished through the linear operator $1-L$, defined by $(1-L)x_i=x_i-x_{i-1}$ [16], where L is the linear backward-shift operator defined by $L^n x_i = x_{i-n}$. Fractional processes [13,14] are obtained by allowing the order ρ in the fractionally differencing operator $(1-L)^\rho$ to take fractional values. After expanding $(1-L)^\rho$ as an infinite binomial series in powers of L, one can show that Eqs. (1a)–(1c) can be expressed as $(1-L)^\rho x_i=\sigma_i\eta_i$.

For $\rho_2\to 0$, all weights become equal, σ_i becomes 1, and process $\mathcal{A}(\rho_1,\rho_2)$ reduces to the ARFIMA process $\mathcal{A}(\rho_1,0)$; for $\rho_1\to 0$, process $\mathcal{A}(\rho_1,\rho_2)$ reduces to the FIARCH process

$\mathcal{A}(0,\rho_2)$; and for positive $\rho_1=\rho_2$, process $\mathcal{A}(\rho_1,\rho_2)$ reduces to process $\mathcal{B}(\rho,\lambda=0)$ proposed in Ref. [17]. Process $\mathcal{B}(\rho,\lambda)$ was proposed with the goal of modeling time series with power-law correlations in $\delta\tau_i$ and an asymmetric distribution of $\delta\tau_i$, and for $\lambda=0$ process $\mathcal{B}(\rho,\lambda)$ generates time series with a symmetric distribution of $\delta\tau_i$ and power-law correlations in $\delta\tau_i$, similar to time series generated by the ARFIMA process $\mathcal{A}(\rho,0)$. Since Eqs. (1a)–(1c) is invariant under the transformation $x_i\rightarrow -x_i$ and $\eta_i\rightarrow -\eta_i$, process $\mathcal{A}(\rho_1,\rho_2)$ generates symmetric probability distributions, i.e., $P(x)=P(-x)$, for arbitrary $\rho_1\in(-0.5,0.5)$ and $\rho_2\in[0,0.5)$.

To quantify the degree of correlations in time series generated by process $\mathcal{A}(\rho_1,\rho_2)$, we employ the method of detrended fluctuation analysis (DFA) [18,19]. In the DFA method one measures the standard deviation $F(n)$ of the detrended fluctuations as a function of the time scale n. If the autocorrelation function $C(n)$ can be approximated by a power law with exponent γ, i.e., if $C(n)\propto n^{-\gamma}$, then $F(n)$ can be approximated by a power law with exponent α, i.e., $F(n)\propto n^{\alpha}$, with $\alpha\approx 1-\gamma/2$ [18]. Hence the value of α represents the degree of correlations in the time series: if $\alpha>0.5$, the time series is power-law correlated; if $\alpha=0.5$, the time series is uncorrelated or short-range correlated; and if $\alpha<0.5$, the time series is power-law anticorrelated.

III. EFFECT OF MODEL PARAMETERS ON CORRELATIONS

Depending on the values of the parameters ρ_1 and ρ_2, process $\mathcal{A}(\rho_1,\rho_2)$ generates different correlation patterns for $\delta\tau_i$ and $|\delta\tau_i|$. For $\rho_1\in(-0.5,0)$, process $\mathcal{A}(\rho_1,\rho_2)$ generates anticorrelated $\delta\tau_i$; for $\rho_1=0$, it generates uncorrelated $\delta\tau_i$, consistent with the FIARCH process $\mathcal{A}(0,\rho_2)$; and for $\rho_1\in(0,0.5)$, it generates correlated $\delta\tau_i$. For $\rho_2=0$ and $\rho_1\in(-0.5,0)$, process $\mathcal{A}(\rho_1,\rho_2)$ generates time series with uncorrelated $|\delta\tau_i|$, consistent with the ARFIMA process $\mathcal{A}(\rho_1,0)$, while for $\rho_2\in(0,0.5)$, it generates time series with correlated $|\delta\tau_i|$ for $\rho_1\in(-0.5,0.5)$. In order to study how correlations in $\delta\tau_i$ and $|\delta\tau_i|$ depend on ρ_1 and ρ_2, we perform numerical simulations of process $\mathcal{A}(\rho_1,\rho_2)$ for varying values of ρ_1 and ρ_2 [20].

First we study correlations in time series generated by process $\mathcal{A}(\rho_1,\rho_2)$ for different values of ρ_1 while keeping ρ_2 fixed. We find from Fig. 1(a) that the variables $\delta\tau_i$ are power-law anticorrelated for $\rho_1\in(-0.5,0)$ and $\rho_2=0.24$. Numerically we find that the exponent $\alpha_{\delta\tau}$ can be approximated by $\alpha_{\delta\tau}\approx 0.5+\rho_1$, which is identical to the relation obtained for the ARFIMA process $\mathcal{A}(\rho_1,0)$. As expected for the integrated signal $\tau_i=\tau_0+\sum_{j=1}^{i}\delta\tau_j$, we obtain that $\alpha_\tau\approx\alpha_{\delta\tau}+1\approx 1.5+\rho_1$. We find from Fig. 1(b) that the magnitudes $|\delta\tau_i|$ are power-law correlated and that all $F(n)$ curves overlap for different values of ρ_1, stating that $F(n)$ and $\alpha_{|\delta\tau|}$ do not depend on ρ_1. In contrast, the time series of $\mathrm{sgn}(\delta\tau_i)$ exhibit short-range anticorrelations that depend on ρ_1. For $\rho_1=0$, Fig. 1 shows that $\delta\tau_i$ and $\mathrm{sgn}(\delta\tau_i)$ are uncorrelated, while $|\delta\tau_i|$ is power-law correlated, consistent with the correlation pattern modeled by the FIARCH process $\mathcal{A}(0,\rho_2)$ and found in finance [15,21].

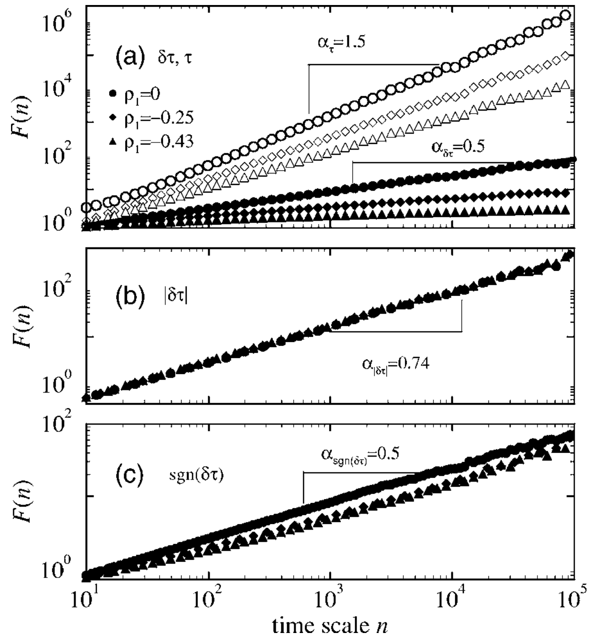

FIG. 1. Detrended fluctuation functions $F(n)$ for time series of 5×10^5 data points generated by process $\mathcal{A}(\rho_1,0.24)$ for varying $\rho_1\in(-0.5,0]$. (a) $F(n)$ for τ_i (open symbols) and $\delta\tau_i$ (closed symbols). For both τ_i and $\delta\tau_i$, each of the $F(n)$ curves can be approximated by a power law $F(n)\propto n^{\alpha}$ with exponents $\alpha_\tau\approx 1.5+\rho_1$ and $\alpha_{\delta\tau}\approx 0.5+\rho_1$. For $\rho_1=0$, the time series $\delta\tau_i$ is uncorrelated ($\alpha_{\delta\tau}\approx 0.5$). (b) $F(n)$ for $|\delta\tau_i|$. For all values of ρ_1, the $F(n)$ curves are virtually identical, indicating that $\alpha_{|\delta\tau|}$ does not depend on ρ_1, and that all $F(n)$ curves can be approximated by a power law $F(n)\propto n^{\alpha}$ with exponent $\alpha_{|\delta\tau|}\approx 0.5+\rho_1$. For $\rho_1=0$, there are power-law correlations in the magnitudes $|\delta\tau_i|$ even though the variables $\delta\tau_i$ are uncorrelated, consistent with the correlation pattern of the FIARCH process. (c) $F(n)$ for $\mathrm{sgn}(\delta\tau_i)$. The $\mathrm{sgn}(\delta\tau_i)$ series show short-range anticorrelations for all values of $\rho_1<0$. For $\rho_1=0$, the $\mathrm{sgn}(\delta\tau_i)$ series is uncorrelated.

Second we study correlations in time series for different values of ρ_2 while keeping ρ_1 fixed. We find from Fig. 2(a) that, separately for $\delta\tau_i$ and τ_i, all $F(n)$ curves overlap, and that they can be approximated by two power laws $F(n)\propto n^{\alpha}$ with $\alpha_{\delta\tau}\approx 0.5+\rho_1$ and $\alpha_\tau\approx 1.5+\rho_1$, stating that the correlations in $\delta\tau_i$ and τ_i do not depend on ρ_2. Figure 2(b) shows that for every ρ_2 the magnitudes $|\delta\tau_i|$ are power-law correlated, and that $\alpha_{|\delta\tau|}$ can be approximated by $\alpha_{|\delta\tau|}\approx 0.5+\rho_2$. We find from Fig. 2(c) that the $F(n)$ curves for the time series of $\mathrm{sgn}(\delta\tau_i)$ overlap for different values of ρ_2, indicating that the correlations in $\mathrm{sgn}(\delta\tau_i)$ do not depend on ρ_2. For $\rho_2=0$, $\delta\tau_i$ are power-law anticorrelated and $\mathrm{sgn}(\delta\tau_i)$ are short-range anticorrelated, while $|\delta\tau_i|$ are uncorrelated, consistent with the correlation pattern generated by the ARFIMA process $\mathcal{A}(\rho_1,0)$ and found in gait dynamics.

The numerical analyses show that, for $\rho_1\in(-0.5,0]$, but not for $\rho_1\in(0,0.5)$, process $\mathcal{A}(\rho_1,\rho_2)$ generates time series with anticorrelations in $\delta\tau_i$ controlled only by ρ_1, and with correlations in $|\delta\tau_i|$ controlled only by ρ_2. Specifically, we find that the exponents α_τ and $\alpha_{\delta\tau}$ depend approximately linearly on ρ_1 through $\alpha_\tau\approx 1.5+\rho_1$ and $\alpha_{\delta\tau}\approx 0.5+\rho_1$, and that the exponent $\alpha_{|\delta\tau|}$ depends approximately linearly on ρ_2

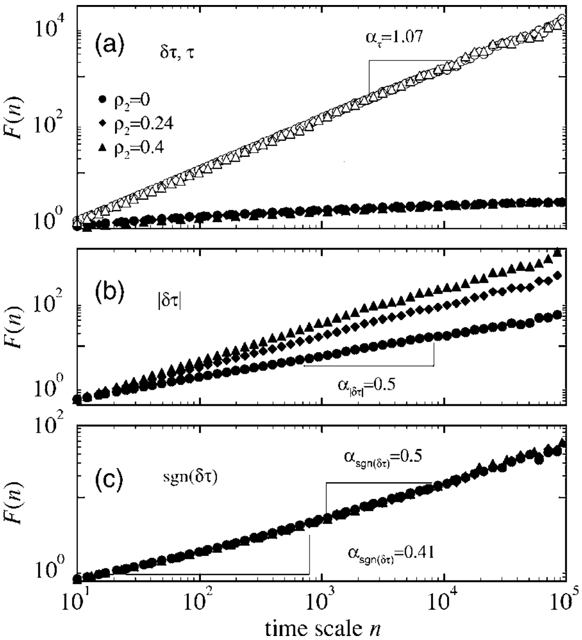

FIG. 2. Detrended fluctuation functions $F(n)$ for time series of 5×10^5 data points generated by process $\mathcal{A}(-0.43,\rho_2)$ for varying $\rho_2\in[0,0.5)$. (a) $F(n)$ for τ_i (open symbols) and $\delta\tau_i$ (closed symbols). For all values of ρ_2, the $F(n)$ curves are virtually identical and can be approximated by a power law $F(n)\propto n^{\alpha}$, indicating that the correlations in τ_i and $\delta\tau_i$ and the exponents $\alpha_\tau\approx1.5+\rho_1$ and $\alpha_{\delta\tau}\approx0.5+\rho_1$ do not depend on ρ_2. (b) $F(n)$ for $|\delta\tau_i|$. Each of the $F(n)$ curves can be approximated by a power law $F(n)\propto n^{\alpha}$ with exponent $\alpha_{|\delta\tau|}\approx0.5+\rho_2$. For $\rho_2=0$, where $\mathcal{A}(\rho_1,\rho_2)$ reduces to the ARFIMA process, the time series exhibits no correlations in $|\delta\tau_i|$, while the variables $\delta\tau_i$ are power-law anticorrelated. (c) $F(n)$ for $\mathrm{sgn}(\delta\tau_i)$. The $\mathrm{sgn}(\delta\tau_i)$ series show short-range anticorrelations, and the $F(n)$ curves are virtually identical, indicating that sign correlations do not depend on ρ_2.

through $\alpha_{|\delta\tau|}\approx0.5+\rho_2$. In the limit of $\rho_1\to-0.5$, process $\mathcal{A}(\rho_1,\rho_2)$ generates correlations with exponent $\alpha_\tau\to1$, which corresponds to $1/f$ noise [1].

For the parameter range $\rho_1\in(0,0.5)$, we find that process $\mathcal{A}(\rho_1,\rho_2)$ generates power-law correlations in both $\delta\tau$ and $|\delta\tau|$. However, we find that the linear relations between $\alpha_{\delta\tau}$ and ρ_1 and between $\alpha_{|\delta\tau|}$ and ρ_2 obtained for negative ρ_1 does not hold anymore for positive ρ_1. Specifically, we find that both parameters ρ_1 and ρ_2 effect each of the exponents $\alpha_{\delta\tau}$ and $\alpha_{|\delta\tau|}$. For the special case of $\rho_1=\rho_2=\rho>0$, we study correlations in $\delta\tau_i$ and $|\delta\tau_i|$ for time series generated by process $\mathcal{A}(\rho,\rho)$. Figure 3(a) shows that, for each value of ρ, the $F(n)$ curves for $\delta\tau_i$ and $|\delta\tau_i|$ can be approximated by power laws with approximately the same exponents $\alpha_{\delta\tau}\approx\alpha_{|\delta\tau|}$, and that $\alpha_{\delta\tau}$ and $\alpha_{|\delta\tau|}$ depend almost linearly on ρ through $\alpha_{\delta\tau}\approx\alpha_{|\delta\tau|}\approx0.5+\rho$.

IV. MAGNITUDE CORRELATIONS AFTER FOURIER PHASE RANDOMIZATION

In this section, we study to which degree magnitude correlations are destroyed by a Fourier phase randomization of the original time series generated by process $\mathcal{A}(\rho_1,\rho_2)$. The

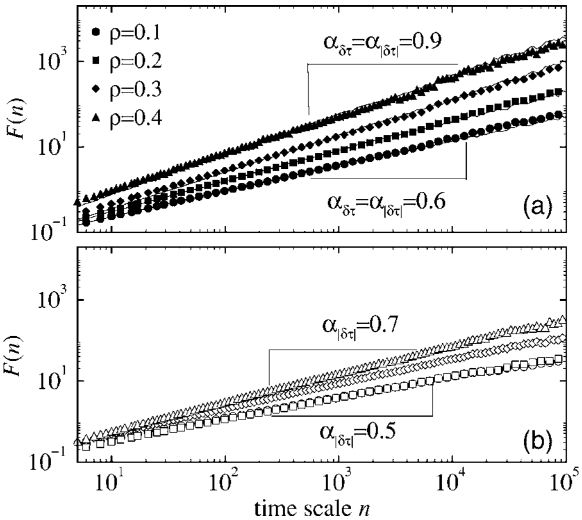

FIG. 3. Detrended fluctuation functions $F(n)$ for a time series of 5×10^5 data points generated by process $\mathcal{A}(\rho,\rho)$ for varying values of $\rho\in(0,0.5)$. (a) $F(n)$ for $\delta\tau_i$ (lines) and $|\delta\tau_i|$ (symbols). For all values of ρ, the $F(n)$ curves for both $\delta\tau_i$ and $|\delta\tau_i|$ are virtually identical, and they can be approximated by a power law $F(n)\propto n^{\alpha}$ with exponent $\alpha_{\delta\tau}\approx\alpha_{|\delta\tau|}\approx0.5+\rho$. (b) $F(n)$ for $|\delta\tilde{\tau}_i|$, where $\delta\tilde{\tau}_i$ denotes the surrogate time series obtained by a Fourier phase randomization of $\delta\tau_i$. For $\rho\lesssim0.2$, magnitude correlations are almost completely destroyed, and for $\rho\gtrsim0.2$, magnitude correlations are reduced, but not completely destroyed, i.e., $\alpha_{|\delta\tilde{\tau}|}<\alpha_{|\delta\tau|}$.

Fourier phase randomization procedure [22] works as follows: perform a Fourier transform of the original time series $\delta\tau_i$, randomize the Fourier phases, but keep the Fourier amplitudes unchanged, and perform an inverse Fourier transform to obtain a surrogate time series $\delta\tilde{\tau}_i$.

First we analyze process $\mathcal{A}(\rho,\rho)$. Figure 3(b) shows that, for each value of ρ, $F(n)$ for the surrogate time series $|\delta\tilde{\tau}_i|$ can be approximated by a power law with exponent $\alpha_{|\delta\tilde{\tau}|}<\alpha_{|\delta\tau|}$. We find that magnitude correlations are almost completely destroyed for $\rho\lesssim0.2$, resulting in an exponent $\alpha_{|\delta\tilde{\tau}|}$ close to 0.5.

Second we study if the ARFIMA process can generate correlations in $|\delta\tau_i|$, and we find that the ARFIMA process shows no correlations in $|\delta\tau_i|$ for $\rho_1\in(-0.5,0)$. However, we find that the ARFIMA process generates magnitude correlations for positive ρ_1. We analyze time series of the ARFIMA process $\mathcal{A}(0.3,0)$, and we obtain from Fig. 4(a) that the ARFIMA process exhibits magnitude correlations that can be approximated by a power law $F(n)\propto n^{\alpha}$ with exponent $\alpha_{|\delta\tau|}<\alpha_{\delta\tau}$. Generally, we find that magnitude correlations become very small for $\rho_1\lesssim0.2$, resulting in an exponent $\alpha_{|\delta\tilde{\tau}|}$ close to 0.5. In contrast to process $\mathcal{A}(\rho,\rho)$, we find from Fig. 4(b) that magnitude correlations of the ARFIMA process remain the same after a Fourier phase randomization.

Third we analyze magnitude correlations of the FIARCH process $\mathcal{A}(0,0.3)$. Figure 4(a) shows power-law correlations in $|\delta\tau_i|$ with exponent $\alpha_{|\delta\tau|}\approx0.5+\rho$, which is in agreement with analytical results from Ref. [15]. Figure 4(b) shows that correlations in $|\delta\tilde{\tau}_i|$ completely disappear after a Fourier phase randomization.

A comparison of Figs. 3(b) and 4(b) shows that the exponent $\alpha_{|\delta\tilde{\tau}|}$ of process $\mathcal{A}(\rho,\rho)$ reduces to the exponent

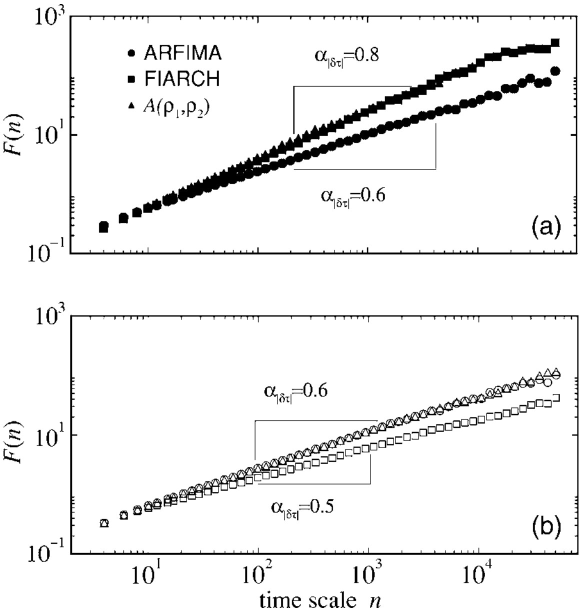

FIG. 4. Detrended fluctuation functions $F(n)$ for time series of 5×10^5 data points generated by the ARFIMA process $\mathcal{A}(0.3,0)$, by the FIARCH process $\mathcal{A}(0,0.3)$, and by process $\mathcal{A}(0.3,0.3)$. (a) $F(n)$ for $|\delta\tau_i|$. For all processes, $F(n)$ can be approximated by power laws with exponent $\alpha_{|\delta\tau|}\approx 0.6$ for the ARFIMA process and with exponent $\alpha_{|\delta\tau|}\approx 0.5+\rho\approx 0.8$ for the FIARCH process and for process $\mathcal{A}(0.3,0.3)$. (b) $F(n)$ for $|\delta\tilde{\tau}_i|$ after a Fourier phase randomization of $\delta\tau_i$. All $F(n)$ curves can be approximated by power laws. For the ARFIMA process, the exponent $\alpha_{|\delta\tilde{\tau}|}$ is equal to the exponent $\alpha_{|\delta\tau|}$. For the FIARCH process, the exponent $\alpha_{|\delta\tilde{\tau}|}$ is reduced to 0.5. For process $\mathcal{A}(0.3,0.3)$, the exponent $\alpha_{|\delta\tilde{\tau}|}$ is reduced to approximately 0.6. It is interesting to observe that $\alpha_{|\delta\tilde{\tau}|}\approx 0.6$ obtained for process $\mathcal{A}(0.3,0.3)$ is approximately equal to $\alpha_{|\delta\tilde{\tau}|}\approx\alpha_{|\delta\tau|}\approx 0.6$ obtained for the ARFIMA process $\mathcal{A}(0.3,0)$.

$\alpha_{|\delta\tilde{\tau}|}\approx\alpha_{|\delta\tau|}$ of the ARFIMA process, stating that magnitude correlations of process $\mathcal{A}(\rho,\rho)$ only partially dissappear. For $\rho_1>0$, both the ARFIMA process and the more general process $\mathcal{A}(\rho_1,\rho_2)$ exhibit magnitude correlations. Hence both processes are good candidates for modeling empirical data with power-law correlations in $\delta\tau_i$ and $|\delta\tau_i|$. However, one fundamental difference between both processes is that the ARFIMA process generates time series whose magnitude correlations are not affected by a Fourier phase randomization of the original time series, whereas process $\mathcal{A}(\rho_1,\rho_2)$ generates time series whose magnitude correlations are partially destroyed by a Fourier phase randomization of the original time series.

V. APPLICATIONS

To test if process $\mathcal{A}(\rho_1,\rho_2)$ might be useful for modeling real-world data, we consider two empirical data sets: heartbeat data with power-law anticorrelations in $\delta\tau_i$ and power-law correlations in $|\delta\tau_i|$, and air temperature data with power-law correlations in $\delta\tau_i$ and $|\delta\tau_i|$ (see Table I).

A. Heartbeat

It is shown in Refs. [6,23–25] that changes $\delta\tau_i$ of heartbeat intervals and their signs exhibit power-law anticorrelations with $\alpha_{\delta\tau}\approx 0.07$ and $\alpha_{\mathrm{sgn}(\delta\tau)}\approx 0.42$, while their magnitudes $|\delta\tau_i|$ are power-law correlated with $\alpha_{|\delta\tau|}\approx 0.74$ and uncorrelated after a Fourier phase randomization of $\delta\tau_i$. Changes of heartbeat intervals as well as changes of physical activity levels, intertrade times in the stock market, and river flux values cannot be modeled by the ARFIMA process because the ARFIMA process generates time series with uncorrelated magnitudes $|\delta\tau_i|$ for $\rho_1<0$, and parameters values $\rho_1<0$ are needed to model anticorrelations in $\delta\tau_i$. In this section we test if process $\mathcal{A}(\rho_1,\rho_2)$ can model power-law anticorrelations in $\delta\tau_i$, power-law correlations in $|\delta\tau_i|$, and short-range anticorrelations in $\mathrm{sgn}(\delta\tau_i)$ comparable to those observed in heartbeat data.

Using the relations $\alpha_{\delta\tau}\approx 0.5+\rho_1$ and $\alpha_{|\delta\tau|}\approx 0.5+\rho_2$ obtained from Figs. 1 and 2, we perform numerical simulations of process $\mathcal{A}(-0.43,0.24)$. Figure 5(a) shows that the simulated time series exhibits power-law anticorrelations in $\delta\tau_i$ and power-law correlations in $|\delta\tau_i|$ with exponents consistent with those observed in heartbeat data [6,23,24]. In addition, we find that the exponent $\alpha_{\mathrm{sgn}(\delta\tau)}\approx 0.42$ from the model time series is similar to the exponent found in heartbeat data. This

TABLE I. Correlation patterns of process $\mathcal{A}(\rho_1,\rho_2)$ for different ranges of ρ_1 and ρ_2. The following abbreviations are used in columns 4, 5, 6, and 10: A ≙ power-law anticorrelated, u ≙ uncorrelated, c ≙ correlated, C ≙ power-law correlated, gait ≙ increments of interstride intervals, finance ≙ price changes, temperature ≙ high-frequency air temperature changes, and several ≙ changes of heartbeat intervals, physical activity levels, river flux values, intertrade times, etc. Underlined results can be derived analytically [13–15], while the remaining results are based on numerical simulations presented in Secs. III and IV. For the case of $\rho_1>0$ and $\rho_2>0$ we could not find simple approximations of $\alpha_{\delta\tau}$ and $\alpha_{|\delta\tau|}$ as a function of ρ_1 and ρ_2.

Process	ρ_1	ρ_2	$\delta\tau$	$\lvert\delta\tau\rvert$	$\lvert\delta\tilde{\tau}\rvert$	$\alpha_{\delta\tau}$	$\alpha_{\lvert\delta\tau\rvert}$	$\alpha_{\lvert\delta\tilde{\tau}\rvert}$	Examples
ARFIMA	<0	0	*A*	u	u	$0.5+\rho_1$	0.5	0.5	gait
White noise	0	0	*u*	*u*	*u*	0.5	0.5	0.5	
ARFIMA	>0	0	*C*	C	C	$0.5+\rho_1$	$<\alpha_{\delta\tau}$	$\alpha_{\lvert\delta\tau\rvert}$	
$\mathcal{A}(\rho_1,\rho_2)$	<0	>0	A	C	u	$0.5+\rho_1$	$0.5+\rho_2$	0.5	several
FIARCH	0	>0	*u*	*C*	u	0.5	$0.5+\rho_2$	0.5	finance
$\mathcal{A}(\rho_1,\rho_2)$	>0	>0	C	C	C	?	?	$<\alpha_{\lvert\delta\tau\rvert}$	
$\mathcal{A}(\rho,\rho)$	>0	>0	C	C	C	$0.5+\rho$	$0.5+\rho$	$<\alpha_{\lvert\delta\tau\rvert}$	temperature

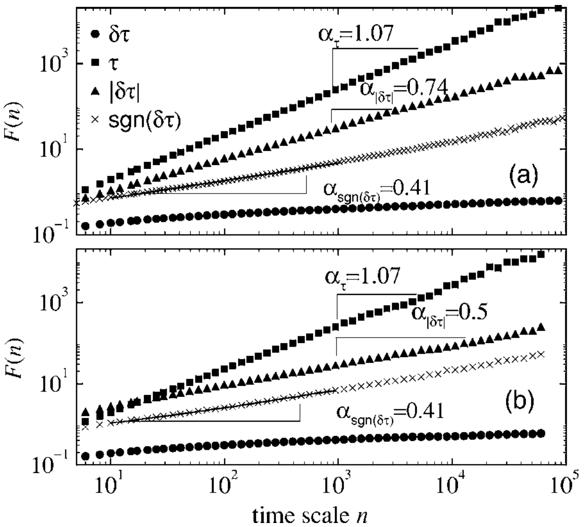

FIG. 5. Detrended fluctuation functions $F(n)$ for time series of 5×10^5 data points generated by process $\mathcal{A}(-0.43,0.24)$ with parameters chosen to simulate heartbeat data. (a) $F(n)$ for $\tau_i, \delta\tau_i, |\delta\tau_i|$, and $\mathrm{sgn}(\delta\tau_i)$. The $F(n)$ curves for $\tau_i, \delta\tau_i$, and $|\delta\tau_i|$ can be approximated by power laws, and the exponents $\alpha_\tau, \alpha_{\delta\tau}$, and $\alpha_{|\delta\tau|}$ are in agreement with empirical heartbeat data [6]. Interestingly, also the $F(n)$ curve for $\mathrm{sgn}(\delta\tau_i)$ is in agreement with empirical heartbeat data [6]. (b) $F(n)$ for $\tilde{\tau}_i, \delta\tilde{\tau}_i, |\delta\tilde{\tau}_i|$, and $\mathrm{sgn}(\delta\tilde{\tau}_i)$ after a Fourier phase randomization of $\delta\tau_i$. $F(n)$ for $|\delta\tilde{\tau}_i|$ can be approximated by a power law with exponent $\alpha_{|\delta\tilde{\tau}|}\approx0.5$, stating that magnitude correlations are completely destroyed. This is consistent with (i) the result of Sec. IV that magnitude correlations of process $\mathcal{A}(\rho_1,\rho_2)$ completely vanish for negative ρ_1 and with (ii) the observation made for empirical heartbeat data [6]. Correlations in $\mathrm{sgn}(\delta\tilde{\tau}_i)$ are virtually identical to correlations in $\mathrm{sgn}(\delta\tau_i)$. Interestingly, this is consistent with the observation made for empirical heartbeat data [6].

is surprising because the parameters ρ_1 and ρ_2 are not chosen to fit the sign correlations, but chosen to fit the observed correlations in $\delta\tau_i$ and $|\delta\tau_i|$.

Figure 5(b) shows that magnitude correlations in the surrogate time series $\delta\tilde{\tau}_i$ are completely destroyed, resulting in an exponent $\alpha_{|\delta\tilde{\tau}|}\approx0.5$, while sign correlations remain unchanged, i.e., $\alpha_{\mathrm{sgn}(\delta\tilde{\tau})}\approx\alpha_{\mathrm{sgn}(\delta\tau)}$. This states that the time series of process $\mathcal{A}(-0.43,0.24)$ exhibits the same correlations in $\delta\tau_i, |\delta\tau_i|, \mathrm{sgn}(\delta\tau_i), |\delta\tilde{\tau}_i|$, and $\mathrm{sgn}(\delta\tilde{\tau}_i)$ as the empirical heartbeat time series.

B. Air temperature

We study air temperature data recorded in 10-min intervals at the Institute of Plant Genetics and Crop Plant Research at Gatersleben [26]. We denote the deseasonalized differences of successive air temperature recordings by $\delta\tau_i$, and in Fig. 6(a) we study short-range correlations in $\delta\tau_i, |\delta\tau_i|$, and $\mathrm{sgn}(\delta\tau_i)$. We find that, for time scales up to 24 h, the air temperature differences are correlated, and that $F(n)$ can be approximated by a power law $F(n)\propto n^\alpha$ with exponent $\alpha_{\delta\tau}\approx0.66>0.5$. We also find from Fig. 6(a) that $|\delta\tau_i|$ and $\mathrm{sgn}(\delta\tau_i)$ are power-law correlated with exponents $\alpha_{|\delta\tau|}\approx0.76$ and $\alpha_{\mathrm{sgn}(\delta\tau)}\approx0.69$.

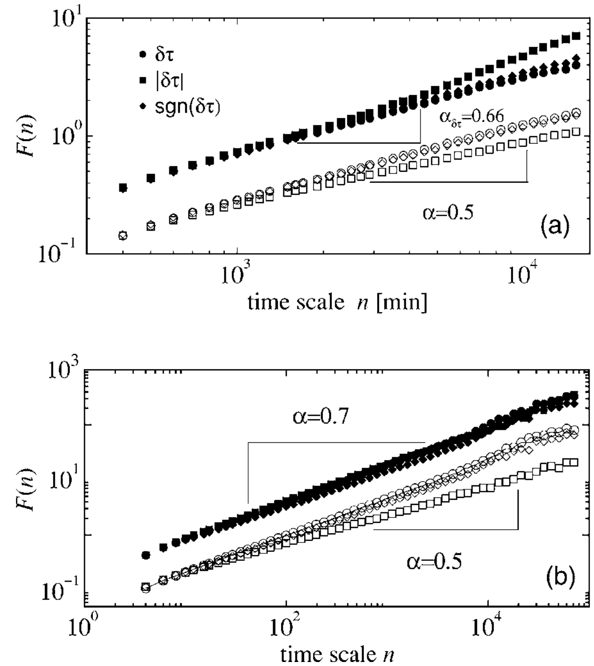

FIG. 6. Detrended fluctuation functions $F(n)$ calculated for deseasonalized air temperature changes and for a time series generated by process $\mathcal{A}(0.2,0.2)$. (a) $F(n)$ for $\delta\tau_i, |\delta\tau_i|$, and $\mathrm{sgn}(\delta\tau_i)$ for empirical air temperature changes before (closed symbols) and after (open symbols) a Fourier phase randomization of $\delta\tau_i$. Each of the $F(n)$ curves for $\delta\tau, |\delta\tau|$, and $\mathrm{sgn}(\delta\tau)$ can be approximated by power laws with exponents $\alpha_{\delta\tau}\approx0.66$, $\alpha_{|\delta\tau|}\approx0.76$, and $\alpha_{\mathrm{sgn}(\delta\tau)}\approx0.69$. After a Fourier phase randomization of $\delta\tau_i$, all $F(n)$ curves can be approximated by power laws with exponents $\alpha_{\delta\tilde{\tau}}\approx0.66$, $\alpha_{|\delta\tilde{\tau}|}\approx0.54$, and $\alpha_{\mathrm{sgn}(\delta\tilde{\tau})}\approx0.61$. This indicates that magnitude correlations of high-frequency air temperature changes $\delta\tau_i$ vanish to some degree, but not completely, by a Fourier phase randomization of $\delta\tau_i$, and that sign correlations remain almost unchanged. (b) $F(n)$ for $\delta\tilde{\tau}_i, |\delta\tilde{\tau}_i|$, and $\mathrm{sgn}(\delta\tilde{\tau}_i)$ generated by process $\mathcal{A}(0.2,0.2)$ before (closed symbols) and after (open symbols) a Fourier phase randomization of $\delta\tau_i$. The $F(n)$ curves for $\delta\tau_i, |\delta\tau_i|$, and $\mathrm{sgn}(\delta\tau_i)$ are almost identical and can be approximated by power laws with exponents $\alpha_{\delta\tau}\approx\alpha_{|\delta\tau|}\approx\alpha_{\mathrm{sgn}(\delta\tau)}\approx0.7$. For the surrogate time series $\delta\tilde{\tau}_i$, the $F(n)$ curves for $|\delta\tilde{\tau}_i|$ and $\mathrm{sgn}(\delta\tilde{\tau}_i)$ can be approximated by power laws with exponents $\alpha_{|\delta\tilde{\tau}|}\approx0.53$ and $\alpha_{\mathrm{sgn}(\delta\tilde{\tau})}\approx0.69$, stating that magnitude correlations are almost completely destroyed by a Fourier phase randomization of $\delta\tau_i$, whereas sign correlations remain almost unchanged.

While changes of air temperature data recorded daily are shown to be power-law anticorrelated [27], changes of air temperature data recorded in 10-min intervals are short-range power-law correlated. This means that on short time scales of the order of minutes an increase of air temperature is followed predominantly by an increase of air temperature, whereas on longer time scales of the order of days an increase of air temperature is followed predominantly by a decrease of air temperature. Interestingly, the exponent $\alpha_{|\delta\tau|}\approx0.76$ calculated for high-frequency air temperature data is only slightly greater than the exponent $\alpha_{|\delta\tau|}\approx0.6$ obtained for air temperature data recorded daily [27].

In the remainder of this section we study if process $\mathcal{A}(\rho_1,\rho_2)$ could be applicable to model empirical time series with power-law correlations in both $\delta\tau_i$ and $|\delta\tau_i|$ similar to those observed in high-frequency air temperature data. We perform numerical simulations of process $\mathcal{A}(0.2,0.2)$, and find from Fig. 6(b) that the time series of $\delta\tau_i, |\delta\tau_i|$, and $\text{sgn}(\delta\tau_i)$ are all power-law correlated with exponents $\alpha_{\delta\tau} \approx \alpha_{|\delta\tau|} \approx \alpha_{\text{sgn}(\delta\tau)} \approx 0.7$, resembling the correlation pattern found in air temperature data.

We study magnitude and signs correlations in the Fourier phase randomized surrogate time series $\delta\tilde{\tau}_i$ of the air temperature changes, and we find from Fig. 6(a) that (i) correlations in $\text{sgn}(\delta\tilde{\tau}_i)$ are similar to correlations in $\text{sgn}(\delta\tau_i)$, and (ii) correlations in $|\delta\tilde{\tau}_i|$ are different from correlations in $|\delta\tau_i|$. Specifically, we find that magnitude correlations are almost completely destroyed.

This behavior implies that the ARFIMA process is not suitable for modeling high-frequency air temperature changes, because magnitude correlations generated by the ARFIMA process are not destroyed by a Fourier phase randomization of $\delta\tau_i$. In the following we study to which degree magnitude correlations of time series $\delta\tau_i$ generated by process $\mathcal{A}(0.2,0.2)$ are destroyed by a Fourier phase randomization of $\delta\tau_i$. Figure 6(b) shows that, after a Fourier phase randomization of $\delta\tau_i$, magnitude correlations almost completely dissappear, and that correlations in $\text{sgn}(\delta\tilde{\tau}_i)$ are equal to correlations in $\text{sgn}(\delta\tau_i)$. This states that correlations in $\delta\tau_i, |\delta\tau_i|, \text{sgn}(\delta\tau_i), |\delta\tilde{\tau}_i|$, and $\text{sgn}(\delta\tilde{\tau}_i)$ observed for high-frequency air temperature data can be modeled by process $\mathcal{A}(0.2,0.2)$.

VI. DISCUSSION AND CONCLUSIONS

We propose a stochastic process $\mathcal{A}(\rho_1,\rho_2)$ that can simulate power-law correlations in both $\delta\tau_i$ and $|\delta\tau_i|$, controlled by only two parameters ρ_1 and ρ_2. For $\rho_1=0$, the process reduces to the FIARCH process $\mathcal{A}(0,\rho_2)$, and for $\rho_2=0$, the process reduces to ARFIMA process $\mathcal{A}(\rho_1,0)$. For the ARFIMA process we find that $|\delta\tau_i|$ are uncorrelated for $\rho_1 \in (-0.5,0)$ and power-law correlated for $\rho_1 \in (0,0.5)$. For $\rho_1 \in (-0.5,0)$ process $\mathcal{A}(\rho_1,\rho_2)$ shows power-law anticorrelations in $\delta\tau_i$, and for $\rho_1 \in (0,0.5)$ and $\rho_2 \neq 0$ it shows power-law correlations in $\delta\tau_i$ and power-law correlations in $|\delta\tau_i|$.

We study to which degree magnitude correlations are changed by a Fourier phase randomization of $\delta\tau_i$, and we find that for the ARFIMA process magnitude correlations remain unchanged, while for the FIARCH process magnitude correlations are completely destroyed. For process $\mathcal{A}(\rho_1,\rho_2)$ magnitude correlations are completely destroyed if $\delta\tau_i$ are anticorrelated, but only partially destroyed if $\delta\tau_i$ are correlated.

We find that the correlation pattern of time series generated by process $\mathcal{A}(-0.43,0.24)$ are in agreement with those found in heartbeat data. Surprisingly, we find that also correlations in $\text{sgn}(\delta\tau_i)$ are in a good agreement with the empirical heartbeat data, and that even $|\delta\tilde{\tau}_i|$ and $\text{sgn}(\delta\tilde{\tau}_i)$ exhibit the same correlations as the corresponding surrogate time series of the empirical heartbeat data.

We analyze air temperature data sampled in 10-min intervals and find that air temperature changes, their magnitudes, and signs are power-law correlated. We find that magnitude correlations almost completely vanish after a Fourier phase randomization of $\delta\tau_i$, whereas sign correlations remain almost unchanged. We find that process $\mathcal{A}(0.2,0.2)$ generates time series with correlations in $\delta\tau_i, |\delta\tau_i|, \text{sgn}(\delta\tau_i), |\delta\tilde{\tau}_i|$, and $\text{sgn}(\delta\tilde{\tau}_i)$ that resemble those obtained in air temperature changes. This is surprising because the parameter $\rho=0.2$ was chosen to approximate correlations in $\delta\tau_i$ and $|\delta\tau_i|$, but not in $\text{sgn}(\delta\tau_i), |\delta\tilde{\tau}_i|$, or $\text{sgn}(\delta\tilde{\tau}_i)$. This surprising agreement might suggest that air temperature changes are possibly driven by a superposition of (i) past values of air temperature changes $\delta\tau_{i-n}$, but not on their magnitudes $|\delta\tau_{i-n}|$, and (ii) a noise term η_i—representing the effect of environmental factors at time i—amplified by a multiplicative factor σ_i that itself depends on the past magnitudes $|\delta\tau_{i-n}|$, but not on past air temperature changes $\delta\tau_{i-n}$.

It is clear that process $\mathcal{A}(\rho_1,\rho_2)$ lacks many important details necessary for modeling heartbeat or air temperature time series, but it might be useful for modeling diverse physical, biological, and social systems exhibiting simultaneously power-law correlations in the variable increments and their magnitudes.

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