PHYSICAL REVIEW A VOLUME 46, NUMBER 4 15 AUGUST 1992

Structural and dynamical properties of long-range correlated percolation

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(Received 21 January 1992)

We develop an algorithm for generating long-range correlations in the percolation problem and investigate their effect on both structural and dynamical properties of the incipient infinite cluster in two dimensions. We find that the fractal dimensions of the backbone and the red bonds (singly connected bonds) are quite different from uncorrelated percolation and vary with λ, the strength of the correlation. Also, we find that the conductivity exponent varies with λ.

PACS number(s): 64.60.Ak

The study of systems possessing spatial disorder has been a very active area of research in recent years. Percolation is a model that has been widely applied to describe the essential physics of such systems [1]. In studies of the percolation model and its variants, spatial disorder has usually been assumed to be uncorrelated—i.e., the probability for any site to be occupied is independent of the occupancy of other sites. However, the nature of disorder in real systems is seldom uncorrelated. For example, the permeability of rock formations is known not to vary randomly in space, but to be consistently high over extended regions of space and low over others; thus it is strongly correlated in space.

The possibility that the elements in a percolation problem experience a long-range spatial correlation has been of long-standing interest [2], but the study of such a correlation has been handicapped by the inability to perform computer simulations. Here we develop an algorithm for generating long-range correlations in the site-occupancy variables of the percolation model. We also investigate the effects of such correlations on the structural and dynamical properties of percolation.

Our first step is to replace the uncorrelated occupancy variables of ordinary site percolation by correlated variables $\{u(\mathbf{r})\}$. If the correlation has a finite range R_0, then the exponents characterizing the correlated system will be unaffected—since if the correlated system is viewed on a scale larger than R_0, it will be uncorrelated [3]. Here we study the effect on the exponents if the correlation has a range that is effectively infinite; such *long-range* spatial correlations are customarily represented by the power-law form

$$\langle u(\mathbf{r})u(\mathbf{r}+\mathbf{R})\rangle \sim R^{-a}. \tag{1}$$

To generate long-range correlated variables [4], we (i) start with random uncorrelated variables $\{w(\mathbf{r})\}$, taken from a uniform distribution; (ii) form the Fourier transform $\tilde{w}(\mathbf{q})\equiv\int w(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r}$; (iii) multiply by a power in q space $\tilde{u}(\mathbf{q})\equiv|\mathbf{q}|^{-\lambda/2}\tilde{w}(\mathbf{q})$; (iv) Fourier transform back to real space, $u(\mathbf{r})\equiv\int\tilde{u}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}$. We thereby obtain long-range correlations in our disordered system, with a correlation function given by

$$g(\mathbf{R})=\langle u(\mathbf{r})u(\mathbf{r}+\mathbf{R})\rangle=\int_{-\infty}^{\infty}|\mathbf{q}|^{-\lambda}e^{-i\mathbf{q}\cdot\mathbf{R}}d\mathbf{q} =f(\lambda)R^{-(d-\lambda)}. \tag{2}$$

The uncorrelated case corresponds to $\lambda=0$ since $f(\lambda=0)=0$ in (2). We can tune the parameter λ ($\lambda<d=2$) to obtain any arbitrary degree of long-range correlation. The case $\lambda<0$ corresponds to antiferro-type correlations, since the site occupancy variables favor a configuration in which the neighbor of an occupied site prefers to be vacant and vice versa; thus the various contributions to the correlation cancel each other faster than they would for an uncorrelated configuration. This results in the correlation decaying with distance even faster than the uncorrelated case.

We place these numbers $\{u(\mathbf{r})\}$ on the sites $\{\mathbf{r}\}$ of a two-dimensional lattice of size L [5]; the variables $\{u\}$ have a Gaussian distribution $P(u)$ which is normalized to the same form for all λ. Next we convert these continuously distributed variables to the corresponding discrete (correlated) occupancy variables given by $\theta(\mathbf{r})=\Theta(\phi-u(\mathbf{r}))$, where ϕ is chosen to give the desired occupied fraction p using the expression $p=\int_{-\infty}^{\phi}P(u)du$.

We next study the dependence on the correlation of various percolation quantities. We begin with the percolation threshold p_c. Since the probability distribution is the same for all λ, any variation in p_c can be attributed entire-

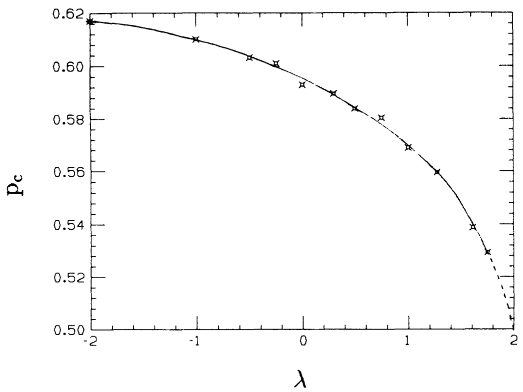

FIG. 1. The phase diagram $p_c(\lambda)$.

ly to the variation in the strength of correlations. We calculate $p_c(\lambda)$ using the Ziff [6] method of measuring the average internal and external cluster perimeters, and identifying p_c as the value of p where the two are equal. We use a lattice of size $L=104$ [5] with 12000 realizations for every λ value. We find that ferro-type correlations $(\lambda>0)$ decrease p_c and antiferro-type correlations $(\lambda<0)$ increase p_c (Fig. 1). For $\lambda\rightarrow 2$, our results extrapolate to $p_c=\frac{1}{2}$, which can be understood from (2) since for $\lambda\rightarrow 2$, the symmetry between vacant and occupied clusters is restored.

To study the effect of long-range spatial correlations on the cluster structure, we calculate [5] the fractal dimension d_f of the incipient infinite cluster and the fractal dimensions d_{BB}, $d_{\min}$, and d_R of its constituent parts, the backbone, the minimum path, and the singly connected "red" bonds, respectively [1]. We find that d_f does not change appreciably [7] with the correlation parameter λ, while the other exponents vary rather dramatically with λ. We find that the backbone gets increasingly compact as λ increases. A quantitative analysis of the tendency, in terms of the values of the various exponents, is presented in Table I and Fig. 2. We see that d_{BB} increases continuously with λ while d_R and $d_{\min}$ decrease. In fact, the cluster appears to become identical to its backbone for $\lambda\gtrsim 1.75$ $(d_f\simeq d_{\mathrm{BB}}\simeq 1.9)$. The variation in all the exponents is consistent with the trend towards increasing compactness as λ increases. In Fig. 3, we plot the values of these exponents as a function of λ and observe that exponents are linear in λ for $\lambda\lesssim 1$, so we can write $d_{\mathrm{BB}}(\lambda)\simeq d_{\mathrm{BB}}(0)+\lambda\Delta$, $d_R(\lambda)\simeq d_R(0)-\lambda\Delta$, and $d_{\min}(\lambda)\simeq d_{\min}(0)-\lambda\Delta/2$. Moreover, the quantities $X_1\equiv d_{\mathrm{BB}}+d_R$, $X_2\equiv d_{\mathrm{BB}}+2d_{\min}$, and $X_3\equiv 2d_{\min}-d_R$ are remarkably λ independent for all values studied.

We anticipate that long-range spatial correlations will have a substantial effect on the *dynamical* exponents, due to the considerable variation in the structural properties of the incipient infinite cluster. Moreover, knowledge of the dynamic properties like the conductance is essential for trying to understand the behavior of real systems, such as the permeability of rock formations. Thus we now consider the conductance exponents μ and $\tilde{\zeta}$ defined by

$$\sigma\sim\begin{cases}(p-p_c)^{\mu}, & L=\infty,\\ L^{-\tilde{\zeta}}, & p=p_c,\end{cases} \tag{3}$$

where σ is the conductance and $\tilde{\zeta}=\mu/\nu$ for $d=2$, with ν being the correlation length exponent. There are many conjectured relations between static and dynamical exponents [1]; these could be more efficiently tested since we have a tunable parameter λ, and all the exponents can be calculated independently as a function of λ.

Here we calculate [5] the conductance exponent $\tilde{\zeta}$ using the following two different methods.

(i) We allow particles to diffuse on the incipient infinite cluster and calculate the mean-square displacement as a function of time [1],

$$\langle r^2\rangle\sim t^{2/d_w}, \tag{4a}$$

where

$$d_w=d_f+\tilde{\zeta}=2+(\mu-\beta)/\nu, \tag{4b}$$

with $d_w\simeq 2.87$ and $\tilde{\zeta}\simeq 0.97$ for uncorrelated percolation. We first calculate d_w using (4a) and then find $\tilde{\zeta}$ using (4b) for various values of λ, and find that $\tilde{\zeta}$ changes substantially with λ [see Figs. 2(c) and 3(d) and Table I].

(ii) We also calculate the conductance directly by reducing the network to a single equivalent conductance using a series of "propagator" transformations [8,9] that in

TABLE I. Dependence upon λ of the various quantities calculated. Studies of the fractal dimensions of various cluster components for q-state Potts droplets [13,14] reveal a remarkable correspondence to the exponents in our correlated percolation problem; namely, for each value of $q\in[1,4]$, we can find a value of $\lambda\in[0,2]$ such that all the corresponding exponents are equal to a good accuracy (compare the last two rows of the table with our values given in rows labeled $\lambda=1.0$ and 1.75, respectively). Consequently, the quantities X_1, X_2, and X_3 defined in the text remain approximately constant as a function of q as well.

λ	p_c	d_{BB}	$d_{\min}$	d_R	$\tilde{\zeta}$
0.0	0.593 ± 0.002	1.61 ± 0.02	1.13 ± 0.01	0.75 ± 0.05	1.02 ± 0.02
0.25	0.590 ± 0.003	1.65 ± 0.03	1.120 ± 0.01	0.71 ± 0.05	0.968 ± 0.02
0.50	0.584 ± 0.003	1.68 ± 0.03	1.113 ± 0.01	0.68 ± 0.05	0.939 ± 0.01
0.75	0.580 ± 0.003	1.73 ± 0.03	1.102 ± 0.01	0.63 ± 0.05	0.860 ± 0.01
1.00	0.569 ± 0.003	1.76 ± 0.03	1.070 ± 0.01	0.60 ± 0.04	0.825 ± 0.01
1.27	0.560 ± 0.004	1.83 ± 0.02	1.048 ± 0.01	0.48 ± 0.05	0.673 ± 0.02
1.75	0.529 ± 0.005	1.90 ± 0.02	1.015 ± 0.01	0.36 ± 0.05	0.396 ± 0.02
$q=2$ Potts [14]		1.75 ± 0.01	1.08 ± 0.01	0.55 ± 0.01	
$q=3$ Potts [14]		1.75 ± 0.02	1.01 ± 0.01	0.32 ± 0.04	

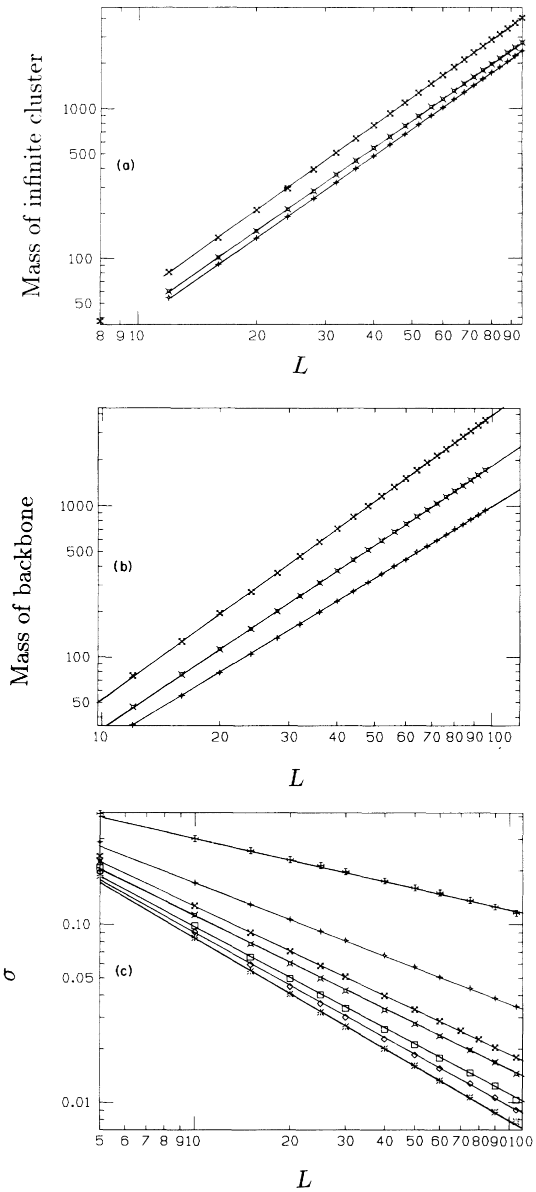

FIG. 2. Analysis used to obtain critical exponents for (a) the fractal dimension of the infinite cluster d_f, (b) the fractal dimension of the backbone d_{BB}, and (c) the conductance exponent $\tilde{\zeta}$. In (a) and (b) $\lambda = 1.75$, 1.0, and 0 reading from top to bottom, while in (c) $\lambda = 1.75$, 1.27, 1.0, 0.75, 0.5, 0.25, and 0 from top to bottom.

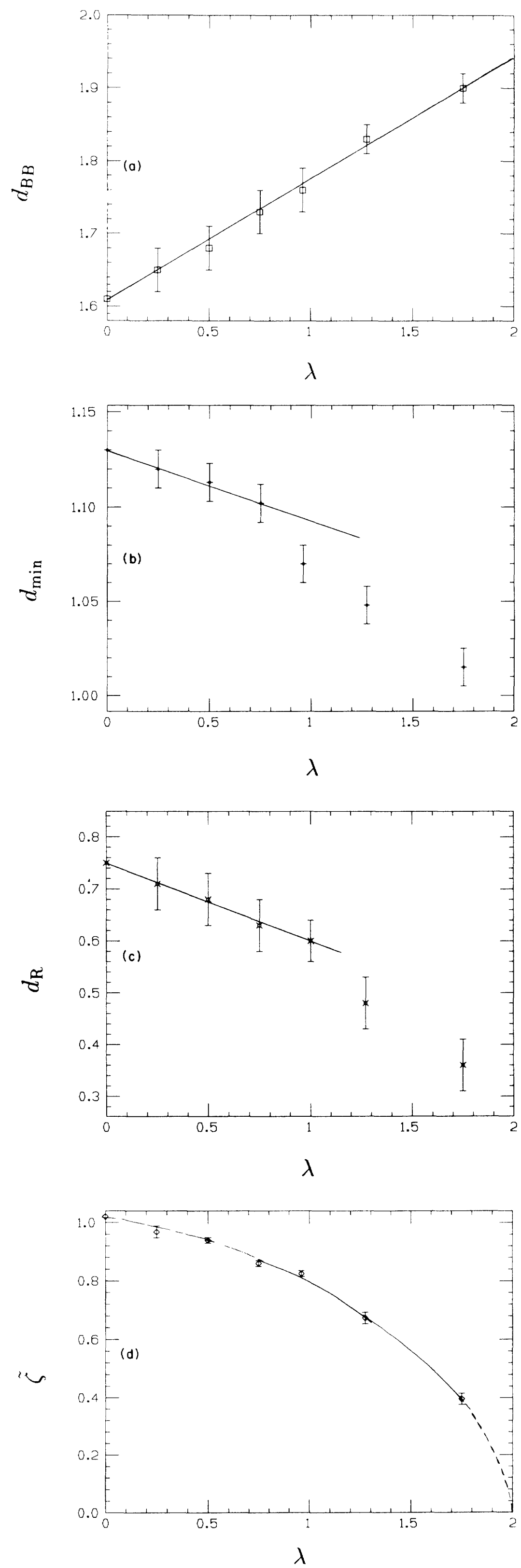

FIG. 3. Variation of exponents with correlation strength λ. (a) Fractal dimension of the backbone d_{BB}. (b) Fractal dimension of the minimum path d_{min}. (c) Fractal dimension of the red bonds d_R. (d) Conductance exponent $\tilde{\zeta}$. Note the trend towards compactness as $\lambda \rightarrow 2$; i.e., $d_{BB} \rightarrow 2$, $d_{min} \rightarrow 1$, $\tilde{\zeta} \rightarrow 0$.

turn are composed of a series of star-triangle transformations. We calculate σ for various lattice sizes and find the exponent $\tilde{\zeta}$ using (3) for several values of λ (Table I). We then compare with the results from diffusion and find that the two sets of values for $\tilde{\zeta}(\lambda)$ agree well with each other, and are consistent with the trend towards increasing compactness with increasing λ [see Fig. 3(d)]. As the correlation gets stronger, the resistance is increasingly dominated by the rare red bonds, and thus $\tilde{\zeta} \rightarrow d_R$ (Table I).

To study the correlation length exponent ν, we first carry out a Monte Carlo renormalization group (RG) calculation, in the spirit of Ref. [10], but with variables that are

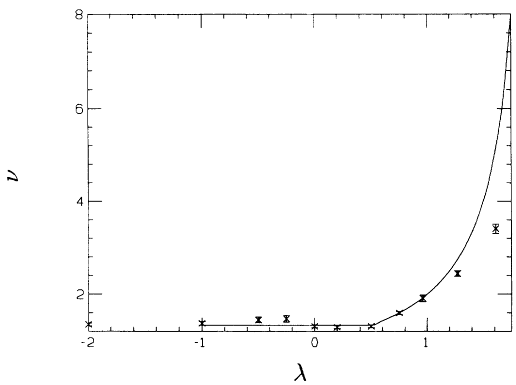

FIG. 4. Correlation length exponent $\nu(\lambda)$.

correlated. To this end, we must find the fixed point $p^*(L)$ of the RG transformation for several values of the lattice size L. From the fixed point we can find the exponent ν as a function of λ by using the scaling relation

$$p^*(L) - p_c \sim L^{-1/\nu}. \qquad (5)$$

We find agreement of $\nu(\lambda)$ with the predictions of Weinrib and Halperin [2] for $\lambda \leq 1.0$ (Fig. 4); specifically, we obtain no change in the exponent from the uncorrelated percolation value for $\lambda \leq 0.5$; for $0.5 \leq \lambda \leq 1.0$, our value agrees with the prediction $\nu = 2/(d-\lambda)$, but for $\lambda \geq 1.0$, our values are consistently lower than this prediction.

Before concluding, we note that several exponents we found *do not* seem to change with the introduction of long-range correlations [7]. These are the fractal dimension d_f of the incipient infinite cluster and exponents related to d_f, the ratio $\beta/\nu, \gamma/\nu$, and the exponent τ of the cluster size distribution [11]. Thus although the universality class changes continuously with λ, strong universality [12] appears to hold in that certain ratios of exponents do not change; this is similar to the situation in a large class of models, including the q-state Potts models [13].

Finally, we compare our results for long-range correlated percolation with the structural properties of Ising clusters at criticality. It is known that the Ising model at criticality possesses correlations of the form (1), i.e.,

$$\langle S_i S_j \rangle \sim R^{-(d-2+\eta)}, \qquad (6)$$

which corresponds to the correlation strength $\lambda = d - \eta = 1.75$ in two dimensions. Consequently, it is interesting to compare our clusters at $\lambda = 1.75$ to the "bare" critical Ising clusters (not Ising droplets [1]). The two systems are identical only up to two-point correlations, moreover even this is true only asymptotically. Thus, we do not expect the exponents to be the same. The percolation threshold for Ising clusters is known to be $p_c = 0.5$ because of symmetry, and we find $p_c = 0.529 \pm 0.005$ for $\lambda = 1.75$. However, we find $p_c \to 0.5$ only as $\lambda \to 2$, also because of symmetry. The results [7] for the Ising case are $d_f = 1.947$ (analytic) and $d_f = 1.90 \pm 0.006$ (simulations). We find $d_f = 1.91 \pm 0.01$ for $\lambda = 1.75$

In summary, the success of our numerical method for generating long-range correlated percolation serves to motivate applications to related problems of interest, such as directed percolation, invasion percolation, and anisotropic percolation. Moreover, since we have a tunable parameter λ, we are also able to conjecture a model independent relation between the various exponents (see the discussion in the caption of Table I).

We thank A. Coniglio, I. Kantor, H. Larralde, C. K. Peng, and D. Stauffer for useful discussions and G. Huber for noticing the similarity between our exponents and those for the q-state Potts droplets. This work was supported by grants from the NSF and the U.S.-Israel Binational Foundation.

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[5] Since we perform operations (i)-(iv) numerically, there is an effective cutoff at both the lower and upper limits of q, due to the finiteness and discreteness of the lattice, respectively. This results in the form (2) of the correlation being valid only up to a certain range $R_{\max}$, which depends on the size of the arrays that are Fourier transformed. We Fourier transform 1024×1024 arrays and obtain $R_{\max} \simeq 100$, and this sets the upper bound on L. The accuracy of our results for exponents is therefore limited by the consequent finite-size effects which explains the slight discrepancy for $\lambda = 0$.

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