Equivalence of the Wilson and Wegner-Houghton generators to first order in perturbation theory at arbitrary anisotropic fixed points*

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We extend our proof of the equivalence of the Wilson and Wegner-Houghton generators (previously given for isotropic fixed points and to first order in a perturbation expansion for the critical exponents) to all fixed points which can be described by the corresponding approximate generators. This includes all those fixed points which are to leading order "wave-vector independent." The proof is again to first order in perturbation theory and exploits the properties of the linearized eigenfunctions (eigenoperators) to evaluate the nonlinear terms.

Recently,[1] we described an approximate differential renormalization-group generator based on the exact Wilson incomplete integration generator.[2] Among the systems studied were the higher-order critical points for isotropically interacting n-component spin systems described by Landau-Ginzberg-Wilson Hamiltonians of degree $\mathcal{O}$ in $\vec{s}^2$. The critical exponents were obtained to first order in the "$\epsilon_{\mathcal{O}}$ expansion" and were shown to agree with the results obtained previously from another approximate generator derived from the Wegner-Houghton equation.[3,4] Independently, Wegner[5] has shown that this equivalence extends to the critical exponents of anisotropic perturbations at these isotropic fixed points. In this note, we extend the proof of this equivalence (to first order) to anisotropic fixed points.

Anisotropic fixed points are of particular interest since many coupled order-parameter problems can be expressed in terms of anisotropic Hamiltonians.[6] For many systems, the fixed-point Hamiltonian is, to leading order, wave-vector independent; that is, it has precisely the simple form of the original Landau-Ginzberg-Wilson initial Hamiltonian. For such systems we can replace the enormous apparatus of the exact renormalization-group formulations with an approximate scheme. For the approximate generators, the wave-vector dependent parts (e.g., gradient terms) are renormalization invariants[1,3] and we do not write them explicitly. The Wilson-based (WB) equation for the Hamiltonian $H(\vec{s}, l)$ is

$$\partial H/\partial l = \lambda_0 H - \Delta\lambda \vec{s}\cdot\vec{\nabla}H + \nabla^2 H - \vec{\nabla}H\cdot\vec{\nabla}H . \tag{1a}$$

The Wegner-Houghton (WH) based generator is

$$\frac{\partial H}{\partial l} = \lambda_0 H - \Delta\lambda \vec{s}\cdot\vec{\nabla}H + \mathrm{Tr}\ln\left(\delta_{ij} + \frac{\partial^2 H}{\partial s_i \partial s_j}\right). \tag{1b}$$

The numbers λ_0 and $\Delta\lambda$ depend on the details of the "gradient" terms. For example, for Lifshitz points,[1,7,8] where the critical propagator takes the form $G^{-1} = \sum |k_i|^{\sigma_i}$, we have $\lambda_0 = \sigma_> \sum d_i/\sigma_i$ (where $\sigma_>$ is the largest of the propagator exponents σ_i) and $\Delta\lambda = \frac{1}{2}(\lambda_0 - \sigma_>)$. This choice is not unique (one may rescale l), but it has the advantage of reducing to $\lambda_0 = d$ and $\Delta\lambda = \frac{1}{2}(d-2)$ if all the propagator exponents $\sigma_i = 2$.

The Gaussian eigenfunctions of WB and WH when linearized around $H = 0$ are polynomials in the spin components s_j; an eigenfunction of degree p has Gaussian eigenvalue $\lambda_0 - p\Delta\lambda$. The eigenfunctions may be orthonormalized with respect to the weight $w(\vec{s}) = \exp(-\frac{1}{2}\Delta\lambda\vec{s}^2)$. If one of the eigenvalues is small, say $\lambda_I \equiv \epsilon \ll 1$, then, in principle, there exist fixed points proportional to $\epsilon\phi_I$, where ϕ_I is some eigenfunction with eigenvalue ϵ. There are generally many such fixed points for each small eigenvalue, since the eigenvalues depend only on the degree of the polynomial. For systems restricted *a priori* to be isotropic, there is only one such fixed point, but for anisotropic systems there are more (some of these may be unphysical in the sense that no initial Hamiltonian can ever reach them). The index I therefore runs over some large set of values which label the eigenfunctions which are appropriate to the system considered.

At the nontrivial fixed point, the Gaussian eigenvalues are corrected by an $O(\epsilon)$ amount, which can be calculated in terms of "triple inner products" of the Gaussian eigenfunctions. We have

$$\lambda_J' = \lambda_J - 2\epsilon\langle I, J|J\rangle/\langle I, I|I\rangle , \tag{2}$$

where the bracket is defined for the WB equation by

$$\langle I, J|K\rangle_{\mathrm{WB}} \equiv \int d^n s\, w(\vec{s})\phi_K \vec{\nabla}\phi_I\cdot\vec{\nabla}\phi_J , \tag{3a}$$

while for the WH equation

$$\langle I,J|K\rangle_{\rm WH}\equiv\frac{1}{4}\int d^n s\, w(\vec{s})\phi_K \times[\nabla^2\vec{\nabla}\phi_I\cdot\vec{\nabla}\phi_J-\vec{\nabla}\phi_I\cdot\vec{\nabla}\nabla^2\phi_J - \vec{\nabla}\phi_J\cdot\vec{\nabla}\nabla^2\phi_I]. \quad (3b)$$

The indices J,K refer to any eigenfunction. Our task is to show that the ratio which appears in (2) is numerically the same whether (3a) or (3b) is used. In Ref. 1, the eigenfunctions ϕ_J were restricted to be isotropic; in that case they are simply the generalized Laguerre polynomials and the proof proceeded via Laguerre polynomial identities and integrations by parts. Here, we must make more direct use of the linear operator.

Using the fact that $\nabla^2\phi_J=\Delta\lambda(\vec{s}\cdot\vec{\nabla}-p_J)\phi_J$ (where p_J is the degree of ϕ_J) we can write (3b) as

$$\langle I,J|K\rangle_{\rm WH}=\frac{1}{4}\int d^n s\, w(\vec{s})\phi_K \times[\nabla^2-\Delta\lambda\vec{s}\cdot\vec{\nabla}+\Delta\lambda(p_I+p_J-2)] \times\vec{\nabla}\phi_I\cdot\vec{\nabla}\phi_J. \quad (4)$$

The differential operator in the square brackets in Eq. (4) is, however, Hermitian with respect to the weight $w(\vec{s})$. Therefore, Eq. (4) simply gives

$$\langle I,J|K\rangle_{\rm WH}=\tfrac{1}{4}\Delta\lambda(p_I+p_J-p_K-2)\langle I,J|K\rangle_{\rm WB}. \quad (5)$$

A similar calculation gives a simple expression for $\langle I,J|K\rangle_{\rm WB}$:

$$\langle I,J|K\rangle_{\rm WB}=\tfrac{1}{2}\Delta\lambda(p_I+p_J-p_K)\int d^n s\, w(\vec{s})\phi_I\phi_J\phi_K. \quad (6)$$

The eigenvalue corrections in (2) depend on $\langle I,J|K\rangle$ only in the case $J=K$. The difference between the WB and WH calculations is only a factor of $\frac{1}{4}\Delta\lambda(p_I-2)$; therefore, the ratio needed for (2) is the same. For explicit calculations, Eq. (6) may be useful.

If ϕ_J belongs to a degenerate set of eigenfunctions, then it is necessary to choose a basis in that subspace such that $\langle I,J|K\rangle=0$ if $J\neq K$. Since $p_J=p_K$, Eqs. (5) and (6) show that this is always possible. In such a basis, Eq. (2) applies directly. If we wish to express matters in a basis independent way, we may say that the corrected eigenspectrum in each degenerate subspace is obtained from the eigenvalues of the matrix $M_{JK}=2\langle I,J|K\rangle/\langle I,I|I\rangle$. In particular, when the fixed point itself is anisotropic, there may be other eigenfunctions with eigenvalue ϵ. The fixed point is stable with respect to these eigenfunctions if $M-I>0$ as a matrix (where J and K index those eigenfunctions degenerate with the fixed-point eigenfunction). This guarantees that the corrected eigenvalues are negative at the fixed point. This is related to the studies of Mukamel and Krinsky[9] who use the exact Wilson recursion relations to study the stability of certain fixed points to $O(\epsilon)$. The result given in this note shows that the approximate WH generator (which is the differential limit of the Wilson recursive procedure) or the simpler WB generator [Eq. (1a)] can be applied to such problems. If $M-I$ has a zero eigenvalue, then one must go to second order in ϵ and the approximate generators may not be adequate; however, in some cases an examination of the global stability relations at $O(\epsilon)$ may still be sufficient to determine the local stability.

A simple example of these criteria is provided by the coupled order parameter problem described in Ref. 6. There are three nontrivial fixed points: the isotropic n-component fixed point, the isotropic $2n$-component fixed point, and a fixed point of mixed symmetry, denoted in Ref. 6 as the "z point." We have

$$(M-I)_{(n)}=\begin{bmatrix}1 & 0\\ 0 & \frac{n-4}{n+8}\end{bmatrix},$$

$$(M-I)_{(z)}=\begin{bmatrix}1 & 0\\ 0 & \frac{(4-n)(n-2)}{n^2+8}\end{bmatrix}, \quad (7)$$

$$(M-I)_{(2n)}=\begin{bmatrix}1 & 0\\ 0 & \frac{2-n}{n+8}\end{bmatrix}.$$

The usual stability conditions are easily read off the matrices in Eq. (7). Similar calculations can be done for the examples of Ref. 9.

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