etc.

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COMMENTS

Nonlinear Crossover between Critical and Tricritical Behavior

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Recently *approximate* solutions to renormalization-group equations were used to calculate the double-power "tricritical" scaling equation of state. We show that this equation can be simply calculated using our *exact* renormalization-group trajectories. The equation of state is expressed *explicitly* in terms of nonlinear scaling fields.

In a recent Letter[1] it was suggested that the crossover or "double-power" scaling functions[2] germane to "tricritical"[3-4] and critical points may be constructed using iteration techniques proposed by Wilson, Nauenberg, and Nienhuis,[5] together with an appropriate mean-field approximation, by matching the Wilson-Fisher critical behavior with a Landau-like expression with fluctuation corrections.

For the problem treated in Ref. 1, approximate solutions of the corresponding recursion relations were derived. On the other hand, we had previously derived[6] the exact $O(\epsilon)$ nonlinear solution of the renormalization group equations for the same problem.[6-8]

In this Comment we show (i) that our exact solutions are particularly convenient for a matching procedure of the sort introduced in Ref. 1, so that there is no necessity to produce approximate solutions, and (ii) that the equation of state[9] can be expressed directly in terms of the nonlinear scaling fields calculated in Ref. 6, permitting a completely *explicit* expression (automatically in asymptotic Griffiths form) for the equation of state, in contrast to the *implicit* equation given in Ref. 1. Our results validate the approach of Ref. 1 and support its claimed potential utility for problems where exact trajectories may not be easily found.

As in Ref. 1 we begin with a Hamiltonian density of the form

$$\mathcal{H} = \tfrac{1}{2}(\nabla\vec{s})^2 - \vec{h}\cdot\vec{s} + \tfrac{1}{2}r\vec{s}^2 + \tfrac{1}{4}u\vec{s}^4 + \tfrac{1}{6}v\vec{s}^6 + \ldots$$

in dimensions $d \equiv 4-\epsilon$, where $\vec{s}(\vec{x})$ is an n-component vector spin field and $\vec{h}$ is the ordering field. The differential equations for r and u [consistent to $O(\epsilon)$] are[6-7,10]

$$\begin{aligned} \mathring{r}_l &= 2r_l + 2(n+2)u_l/(1+r_l), \\ \dot{u}_l &= \epsilon u_l - 2(n+8)u_l^2/(1+r_l)^2, \end{aligned} \tag{1}$$

where the fluxions denote differentiation with respect to the renormalization parameter l. The approximate solutions of Ref. 1 were obtained to leading order in $u(l)$ and ϵ by assuming that $|r_l| \lesssim 1$. However, for r and u noncritical, $|r_l| \to \infty$ and $u_l \to \infty$ under the renormalization group action.[11,12] The approximation $|r_l| \lesssim 1$ was avoided in Ref. 6, where it was necessary to assume only that the *critical* values of r and u are $O(\epsilon)$.

Reference 1 uses a modified homogeneity relation for the free energy,[5,7,13,14]

$$F(h,r,u) = e^{-dl}F(h_l, r_l, u_l) + n\int_0^l e^{-dl'} \ln(1+r_{l'})\,dl', \tag{2}$$

where $h_l \equiv h\exp[(1+d/2)l]$. Reference 1 suggests that for large l, the first term be treated in the mean-field approximation (with fluctuation corrections). Because of their approximations, the limit $l\to\infty$ cannot be taken. Instead, they use a value of l such that $|r_l| \approx 1$. *Using the exact solutions, we may take the $l=\infty$ limit.* We find that $r_l\exp(-2l)$ and $u_l\exp(-\epsilon l)$ have finite limits as $l\to\infty$. Using these limiting forms[15] we are able to calculate the magnetization equation of state [since the second term of (2) does not contribute to the equation of state, it will not be discussed further here[16]]. The general form of the equation of state obtained conforms to that suggested for the global behavior of competing fixed points.[7] From (2) we obtain an explicit expression for the renormalized mean-field approximation to the equation of state:

$$h = \lim_{l\to\infty}[r_l\exp(-2l)M + u_l\exp(-\epsilon l)M^3]. \tag{3}$$

To calculate $r_l\exp(-2l)$ and $u_l\exp(-\epsilon l)$, we use the nonlinear scaling fields derived in Ref. 6, which are given in terms of variables

$$\begin{aligned} t_l &\equiv \frac{r_l}{1+r_l} + (n+2)\frac{r_l}{(1+r_l)^2}, \\ y_l &\equiv \frac{1}{2\epsilon(n+8)}\frac{u_l}{(1+r_l)^2}. \end{aligned} \tag{4}$$

The Wilson-Fisher fixed point is at $t=0$, $y=1$, while the infinite Gaussian point is at $t=1$, $y=0$.

The nonlinear scaling fields are[6]

$$\begin{aligned} (1+r_l)t_l/Y_l^{\Delta_n} &= \text{const}\, e^{2l}, \\ u_l/Y_l &= \text{const}\, e^{\epsilon l}, \end{aligned} \tag{5}$$

where $\Delta_n \equiv (n+2)/(n+8)$ and

$$Y_l \equiv (1 - y_l/\varphi_l)\exp(\epsilon t_l \Delta_n y_l/\varphi_l)$$

with

$$\varphi_l \equiv |1-t_l|^{d/2}\exp\tfrac{1}{2}\epsilon t_l(1-2\Delta_n).$$

Thus

$$\begin{aligned} r_l\exp(-2l) &= \frac{(1+r)t}{Y^{\Delta_n}}\left[\frac{r_l Y_l^{\Delta_n}}{(1+r_l)t_l}\right], \\ u_l\exp(-\epsilon l) &= \frac{u}{Y}Y_l, \end{aligned} \tag{6}$$

where henceforth we write r, u, t, and Y for r_0, u_0, t_0, and Y_0. For all noncritical Hamiltonians, $|r_l|\to\infty$ (and $t_l\to 1$) as $l\to\infty$. Thus, we need only calculate Y_l for large l. We expect Y_l to approach some invariant limit and therefore Y_∞ should be expressible in terms of the renormalization invariant[6]

$$I \equiv \frac{|t_l|^\epsilon Y_l^{2-\epsilon\Delta_n}}{|1+r_l|^d y_l^2} \xrightarrow[\text{large } l]{} \frac{Y_l^{2-\epsilon\Delta_n}}{|r_l|^d y_l^2}. \tag{7}$$

We see immediately that if Y_∞ is to be finite and nonzero, we must have $|r_l|^d y_l^2$ invariant for large l. To the required order, we find $Y_\infty = I^\nu/(I^\nu+1)$, where $\nu^{-1} = 2-\epsilon\Delta_n$. Therefore we can write

$$\lim_{l\to\infty} r_l\exp(-2l) = \frac{(1+r)t}{Y^{\Delta_n}}Y_\infty^{\Delta_n}, \tag{8a}$$

$$\lim_{l\to\infty} u_l\exp(-\epsilon l) = (u/Y)Y_\infty. \tag{8b}$$

Combining (8) with (3) we obtain

$$h = Mt(1+r)\left(\frac{Y_\infty}{Y}\right)^{\Delta_n} + M^3 u\frac{Y_\infty}{Y}. \tag{9}$$

In (9) the nonlinear crossover information is contained for *all* values of t, not just in the critical region ($t\ll 1$). Since the behavior for *large t* of the equation of state should not be expected to be universal, we will use the forms of (9) valid for $t\ll 1$ (retaining, of course, any singular term for $t\to 0$). Consonant with the approximations already made, we will take the $O(\epsilon^0)$ parts of (9), retaining ϵ only in the exponents. On making the changes of scale $h\to h[\epsilon(n+8)]^{-1/2}$ and $M\to M[2\epsilon(n$

$+8)]^{1/2}$, (9) becomes

$$h = Mt\left(\frac{|t|^{\epsilon\nu}}{y^{\gamma} + |t|^{\epsilon\nu}(1-y)}\right)^{\Delta_n} + M^3 y \frac{|t|^{\epsilon\nu}}{y^{\gamma} + |t|^{\epsilon\nu}(1-y)}, \qquad (10)$$

where $\gamma = 2\nu$. Of course (10) represents only the lowest-order approximation to the equation of state. However, it can be shown[11] that even the higher-order corrections are functions of the nonlinear scaling fields given in (8). Thus although the details and accuracy of the equation of state are changed by the higher-order terms, the nonlinear crossover effects are the same.

Equation (10) explicitly exhibits the double-power structure of nonlinear crossover between the critical and "tricritical" behavior.[2,7] Various limiting behaviors of (10) are easily obtained:

(i) *$t \to 0$ with y held fixed.*—The equation of state is

$$h = M\frac{t|t|^{\epsilon\nu\Delta_n}}{y^{\gamma\Delta_n}} + M^3\frac{|t|^{\epsilon\nu}}{y^{\gamma-1}}. \qquad (11a)$$

Thus, we immediately see that $\gamma = 1 + \epsilon\Delta_n/2$ *is* the susceptibility exponent and that $2\beta = 1 + \frac{1}{2}\epsilon(\Delta_n - 1)$ *is* the magnetization exponent. Dividing through by $M^{\delta} = M^{3+\epsilon}$, we see that (11a) is in the usual Griffiths asymptotic scaling form[9,17]:

$$\frac{h}{M^{\delta}} = \frac{\operatorname{sgn} t}{y^{\gamma\Delta_n}}\left(\frac{|t|}{M^{1/\beta}}\right)^{\gamma} + \frac{1}{y^{\gamma-1}}\left(\frac{|t|}{M^{1/\beta}}\right)^{\gamma-2\beta}. \qquad (11b)$$

(ii) *$y \to 0$ with $y^{\gamma}|t|^{\epsilon\nu}$ fixed.*—Y_{∞} is simply a constant, and we obtain the equation of state for the Gaussian model,

$$h = Y_{\infty}Mt + Y_{\infty}^{\Delta_n}M^3|t|^{\epsilon/2}. \qquad (12a)$$

Making additional scale changes in M and h, we write (12a) in scaling form:

$$\frac{h}{M^{\delta}} = \left(\frac{t}{M^{1/\beta}}\right)^{\gamma} + \left(\frac{t}{M^{1/\beta}}\right)^{\gamma-2\beta}, \qquad (12b)$$

with $\gamma = 1$ and $2\beta = 1 - \epsilon/2$.

The method employed here is general. For each Hamiltonian, we need to calculate the limits of the renormalized coupling constants for large l. If u_m is a coupling constant for a term involving m spins, we need $\lim_{l\to\infty} u_m(l)\exp\{-[d + m(2-d)]l\}$. This always is given by a simple nonlinear scaling field multiplying some function of the nonlinear renormalization invariants.

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[8]The exact $O(\epsilon)$nonlinear solutions of the renormalization-group equations describe the crossover between critical and Gaussian behavior for an isotropic n-vector model. The Gaussian fixed point may be considered as describing the tricritical behavior for $d \geq 3$ (cf. Ref. 3). However, a strict perturbational analysis would require r, u, and v to be of the same order with $v > 0$; the feedback from the eight-spin term must also be considered (cf. Ref. 4). The conventional tricritical systems, with Hamiltonians like $As^2 + Bs^4 + s^6$, with B vanishing at the tricritical point, have mean-field (not Gaussian) behavior for $d > 3$. The Gaussian fixed point may also be viewed as a nonclassical tricritical fixed point for the type discussed recently in Ref. 7.

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[12]For $T < T_c$, we have $r_l \to -\infty$, leading to apparent anomalies in (1) and (2). This difficulty is not just a breakdown of the particular equations, but rather a fail-

ure of perturbation theory itself at the point $r_l = -1$. A nonperturbational view (cf. Ref. 11) shows that this anomaly is entirely artificial. Thus it is not surprising that it can be removed in several ways (one method is the "spin-shift" approach used in Ref. 1). Actually, we need only define some continuation of the trajectories through $r_l = -1$; the spin shift is not necessary. For the magnetization equation of state, the expression for $T > T_c$ can be simply continued to $T < T_c$; i.e., Eq. (10) holds for $t > 0$ *and* for $t < 0$ (cf. Ref. 9b, which illustrates this for the Brézin-Wallace-Wilson result). This corresponds to the fact that in field-theoretic calculations, the temperature can be treated as an s^2 insertion rather than being incorporated into the propagator (cf. Ref. 9b).

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[14]As in Ref. 1, this form is only convenient for $T > T_c$. We use it illustratively for simplicity. As discussed in Refs. 11 and 12, the case $T < T_c$ presents no real difficulty.

[15]These limits define the "dressed mass" $m^2 \equiv \lim_{l\to\infty} r_l \times \exp(-2l)$ and renormalized four-spin coupling constant $u_r \equiv \lim_{l\to\infty} u_l \exp(-\epsilon l)$; cf. J. Rudnick, Phys. Rev. B 11, 363 (1975).

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ERRATUM

EVIDENCE FOR GIANT $M2$ STATES IN ^{208}Pb. R. A. Lindgren, W. L. Bendel, L. W. Fagg, and E. C. Jones, Jr. [Phys. Rev. Lett. 35, 1423 (1975)].

In Fig. 1 the units of $d^2\sigma/d\theta dE$ should be μb/sr MeV (not nb/sr MeV) on both vertical scales.