Global features of nonlinear renormalization-group equations*

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The analysis of selected nonlinear problems in the renormalization group is found to show striking contrasts between the usual local linearized fixed-point analysis and the properties of global solutions of nonlinear equations derived from an approximation of the Wegner-Houghton differential formulation. The competition between various fixed points that is incorporated in *general* global solutions can upset the asymptotically valid critical behavior deduced from the local analysis. In general, the critical-point exponents of such a solution will not satisfy equalities, but rather the corresponding inequalities. However, these nonscaling solutions have extraneous singularities that are not related to the thermodynamic singularities of the system. If singularities of this type are excluded, then the global solution has the same critical-point exponents as the local solution derived by linearizing around the stablest fixed point. It is shown that in this case the critical surface in the Hamiltonian space is closely related to the surface of order-2 critical points in a thermodynamic field space. The boundaries of this surface are correspondingly related to the critical points of higher order in this thermodynamic space. The nonlinear global solution predicts multiple power scaling behavior from a single scaling equation deduced from the renormalization group. Previously such behavior was obtained by *postulating the simultaneous validity of two of more "linear" scaling hypotheses*.

I. INTRODUCTION

The renormalization-group approach to the study of critical phenomena is a mathematical expression of certain heuristic ideas of Kadanoff.[1] Kadanoff argued that sufficiently near the critical point, the correlation length was so large that even crude averages over small groups of spins would not alter the physics in an unmanageable way, but would only change the parameters slightly. If the transformation of the parameters is assumed to be of a particular form (the "scaling hypothesis"), then many valid and useful predictions of critical behavior follow.[1,2] In the renormalization-group approach,[3-5] a particular form of Kadanoff averaging is carried out explicitly. If the system Hamiltonian is characterized by some set of parameters $\{p_i\}$, the renormalization-group equations provide a definite transformation on the parameter space.

The fixed points of the renormalization group are just the fixed points of this transformation in the parameter space. As is well known from the study of nonlinear finite-difference and differential equations,[6] the qualitative and much of the quantitative properties of a set of transformations are determined by the location and study of the fixed points of those transformations. This is the rationale of the renormalization-group approach: to study the transformation properties (via the fixed points) in order to deduce the properties of the partition function and other thermodynamic quantities.

A formulation of a renormalization group may be of a recursive character with a "finite-difference" generator, or it may have a differential generator. For example, if we consider a system with discrete spins localized on lattice sites, we could construct a renormalization group which replaced each spin by an average of that spin and the spins of its neighbors.[7] After averaging, the parameters of the Hamiltonian would, in general, change. The new parameters would be given by relations of the form $p'_j = P_j(\{p_i\})$ for some functions P_j. Thus the renormalization-group equations in the parameter space would take the form of finite-difference equations coupling all the parameters together. A second case of finite-difference formulation is the well-known renormalization group of Wilson.[5] It treats a system of continuum spins; the renormalization average is performed by averaging over a finite fraction[8] of the momenta in the space of the Fourier transform of the spin density.

Finite-difference equations, however, are clumsy to manipulate in the large, i.e., over large domains of the variables. A differential generator, which performs an average over an infinitesimal number of degrees of freedom, is far more convenient. A differential generator gives a smooth transformation of the p_j, of the form $dp_j/dl = P_j(\{p_i\})$, where l is a parameter describing the progress of the renormalization averaging. Various differential generators have been proposed; e.g., Wilson[5] has proposed a "partial-integration" generator, while Wegner and Houghton[9] have proposed a differential generator which averages over an infinitesimal shell of momentum.

In applications of the renormalization group to critical phenomena, it is customary to perform an average which corresponds to a simple scale change of the correlation length as in Kadanoff scaling. For a *finite-difference* generator we expect that the

renormalization equation is of the form $\xi_{l+1} = \text{const}\ \xi_l$. For a *differential* generator, the parameter l is usually normalized so that the renormalization equation for ξ is $\dot{\xi} = -\xi$, where the dot denotes differentiation with respect to l.

With a few exceptions,[10,11] the work devoted to the application of the renormalization group to critical phenomena has been confined to the location and linearized analysis of fixed points. For example, we could consider a set of two parameters p and q with renormalization group equations

$$\dot{p} = 2p[1 - p - \epsilon(\tfrac{1}{2}\Delta)q], \tag{1.1a}$$

$$\dot{q} = q[\epsilon(1 - q) - 4p]. \tag{1.1b}$$

These equations have several fixed points. One fixed point is at $p = q = 0$. If we *linearize* around $p = q = 0$ we obtain the elementary solutions $p = p_0 e^{2l}$ and $q = q_0 e^{\epsilon l}$.

In terms of p and q, the equation for the correlation length $\dot{\xi} = -\xi$ becomes

$$\frac{\partial \xi}{\partial p}\dot{p} + \frac{\partial \xi}{\partial q}\dot{q} = -\xi. \tag{1.2}$$

If we make the linearized approximation for $\dot{p}$ and $\dot{q}$, (1.2) is of the form

$$\sum_i a_i p_i \frac{\partial F}{\partial p_i} = a_F F. \tag{1.3}$$

The solutions of an equation such as (1.3) are generalized homogeneous functions (GHF's).[12] That is, they satisfy the functional relationship

$$F(\{\lambda^{a_i} p_i\}) = \lambda^{a_F} F(\{p_i\}). \tag{1.4}$$

The constants a_i and a_F are termed the scaling powers of the variables p_i and the function F, respectively.

To see that (1.3) implies (1.4) it is sufficient to examine the case at hand. We write

$$\xi(p_0 e^{2l}, q_0 e^{\epsilon l}) = \xi(p_0, q_0) e^{-l}, \tag{1.5}$$

which is just (1.4) with $\lambda = e^l$, $a_p = 2$, $a_q = \epsilon$, and $a_\xi = -1$. Equation (1.5) is equivalent to

$$\xi(p, q) = |p|^{-1/2} \xi(\text{sgn}p, q/|p|^{\epsilon/2}). \tag{1.6}$$

The quantity $q/|p|^{\epsilon/2}$ is a *renormalized invariant* of the linearized equations, as is easily checked from (1.1).

The correspondence between the form for the correlation length and the usual scaling hypothesis leads to the definition of p and q as (*linear*) *scaling fields*. If we make the identification $p \sim T - T_c$, we derive the value of the critical-point exponent $\nu = \frac{1}{2}$. However, we linearized (1.1) to obtain this solution. In principle this analysis might only be valid *locally*, infinitesimally close to the fixed point $p = 0$, $q = 0$ (cf. Fig. 1).

We can examine other fixed points. A second fixed point is located at $p = 0$, $q = 1$. At this fixed point, we have a different pair of *linear* scaling fields, $p' \equiv p$ and $q' \equiv (q - 1) + [4/(2 - \epsilon\Delta)]p$, with the new *linearized* renormalization equations:

$$\dot{p}' = (2 - \epsilon\Delta)p', \tag{1.7a}$$

$$\dot{q}' = -\epsilon q'. \tag{1.7b}$$

We again obtain a GHF but with variables p' and q' and scaling powers $a_{p'} = 2 - \epsilon\Delta$ and $a_{q'} = -\epsilon$. Thus,

$$\xi(\lambda^{2-\epsilon\Delta}p', \lambda^{-\epsilon}q') = \lambda^{-1}\xi(p', q'), \tag{1.8a}$$

or equivalently

$$\xi(p', q') = |p'|^{-1/(2-\epsilon\Delta)} \xi(\text{sgn}p', q'|p'|^{\epsilon/(2-\epsilon\Delta)}). \tag{1.8b}$$

Again, since the fundamental equations (1.1) were linearized, the solution given in (1.8) is, in principle, valid only infinitesimally near the fixed point $p = 0$, $q = 1$.

Thus, by locating *two* fixed points and analyzing the behavior of the linearized equations in a neighborhood of each fixed point we have produced two *competing* forms for the correlation length with *different critical exponents*, $\nu = \frac{1}{2}$ and $\nu' = 1/(2 - \epsilon\Delta)$.

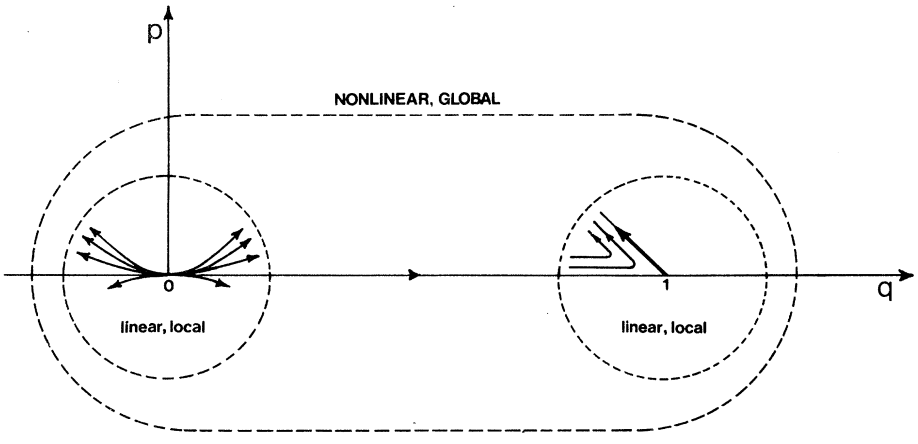

FIG. 1. (p, q) plane of Eq. (1.1). Only two fixed points are shown. Local integral curves for the fixed points (0, 0) and (0, 1) are sketched as shown. The local regions of validity of the linearized approximations to the correlation length [cf. Eq. (1.6) and (1.8)] are indicated, as well as the nonlinear global region considered in Sec. II.

If we are not to be confined to infinitesimal regions about *one* of the fixed points, the effects of *both* fixed points must be incorporated. This obviously requires a solution of the *nonlinear equations* to give a solution valid at each fixed point and at every point between the two fixed points. To include more than a single fixed point, the *local linear* analysis must be replaced by a *global nonlinear* analysis (cf. Fig. 1).

From the example treated above it is easy to see that any set of *linearized* renormalization-group equations confirm the scaling ideas of Kadanoff: Thermodynamic functions are GHF's of suitable *linear* combinations of the parameters $\{p_i\}$. Since many fixed points may be included in a global analysis, three questions must be answered:

(i) Which fixed point should be chosen to represent the true scaling behavior of the system?

(ii) Can solutions derived from linearizing the renormalization-group equations around various fixed points be matched together in such a way as to form a globally valid solution?

(iii) Does the class of global solutions include behavior that is drastically different than the behavior deduced from the linearized solutions?

Question (i) has been traditionally answered by the criterion of *relative stability*. If two fixed points can be considered as important for a particular system, we examine them to determine whether a trajectory in the parameter space connects them. If such a path exists, and under the action of the renormalization equations it passes from fixed point A to fixed point B, we say that A is unstable with respect to B. It is assumed that the fixed point which is least unstable (of those fixed points which lie on the "critical surface," cf. Sec. III) is the dominant or controlling fixed point. The asymptotically valid scaling behavior is assumed to be that given by the linearization about that point. For example, the isotropic Heisenberg fixed point is unstable with respect to an anisotropy along one spin axis. In the renormalization-group parameter space, two paths lead out of the Heisenberg point, connecting it to a point of XY character and a point of Ising character.[13] We see that the idea that the slightest bit of anisotropy turns the system into either an XY-like or Ising-like system is supported by this notion of relative stability (and seems to be confirmed by high-temperature series analysis).[14] The procedure of checking relative stability requires first *finding* the fixed points of the renormalization-group transformations and this is an exceedingly nontrivial task. Numerous perturbation expansions have been developed to discover those fixed points that are "close" to some fixed point located by inspection (the ϵ expansions of Refs. 4 and 5 and the ϵ_0 expansions of critical points of higher "order" of Refs. 15 and 16 are examples).

Question (ii) must be answered individually for each renormalization group and probably for each problem within any one renormalization group. At least for groups with differential generators, one supposes that the solutions for the thermodynamic functions are again GHF's with revised arguments. That is, instead of *linear* combinations of the parameters $\{p_i\}$, certain *nonlinear* functions of the parameters (called *nonlinear scaling fields*)[17] will be the arguments of the GHF. The equations for the nonlinear scaling fields will be first-order partial-differential equations with coefficients that are nonlinear in the p_j. For any particular case, the general theory of such equations can be invoked to determine whether solutions to these equations exist in the large.[18] A further question is whether every global solution for the thermodynamic function matches onto the local solutions at all the fixed points (or at least at all the fixed points we have found); that is, are all the global solutions sufficiently regular (in a sense particular to each problem) near each fixed point? (Of course, the nonlinear solution for the $\{p_i\}$ themselves always match.) In general, the answer is no; *not every global solution matches onto the linearized solution at each fixed point*. Many global solutions exist that have singularities that are unrelated to the physical thermodynamic singularities.[19] If we require that a global solution match smoothly at each fixed point, then the set of global solutions will be restricted, but the global solution is still not in general uniquely determined.

Question (iii) can only be answered by explicit construction of the nonlinear scaling fields and some class of global solutions. In this work we will consider two cases which illustrate that the answer is yes: Global solutions can be radically different than what might be presumed from the local analysis. However, we will also show that the global solutions that violate the local analysis have extra singularities on the boundary or some portion of the boundary of the solution region. These singularities are apparently unrelated to usual thermodynamic singularities. *If we require that the solution be well behaved everywhere on the boundary of the solution region*, then the only global solutions that are acceptable support the local linearized analysis.

In Sec. II we review the nonlinear solution given in Ref. 11 for the crossover or competition between the Gaussian fixed point (which has Gaussian exponents) and the Wilson-Fisher fixed point (which has non-Gaussian exponents). We show that a general global solution is not dominated by the stabler fixed point. Such solutions, however, have singularities on the separatrix which emerges from the stabler fixed point. This separatrix also forms part of the boundary of the solution region. The

exclusion of those solutions with singularities on the separatrix leaves only solutions which are dominated by the local linearized behavior of the stabler fixed point.

In Sec. III we discuss a three-parameter crossover problem. The system considered consists of two internally isotropic n-spin subsystems which are coupled together through a biquadratic term. The competition is among a fixed point of dubious spin and the usual Gaussian, n-spin, and $2n$-spin fixed points. In this case it is again true that a global solution which is not dominated by the linear behavior of the stablest fixed point has singularities on the boundary of the solution region. However, the singularities do not cover the entire bounding surface (which is two dimensional) but are confined to the *line* emerging from the stablest fixed point. The removal of the singularity along this separatrix again restricts the class of global solutions to those dominated by the behavior of the stablest fixed point. The properties of these restricted global solutions strongly resemble the crossover behavior of systems which contain several different types of critical points, including critical points of higher order.[20] In particular, the "double-power-law" scaling behavior characteristic of critical to tricritical crossover[21] is an automatic consequence of the nonlinear renormalization-group solutions.

In Sec. IV we discuss the general properties of global renormalization-group solutions as illustrated in Secs. II and III. We discuss the possibility of accepting the global solutions which are singular on some separatrix. The critical-point exponents of such systems are more complicated than the more regular global solutions, but are still characterized by double-power-law expressions. These systems, which are a generalization of scaling systems, share many properties with the simpler systems; elsewhere, we have given a partial classification and discussion of such systems and termed them "critically ordered" systems.[22]

II. TWO-PARAMETER CROSSOVER

In this section we review the crossover solution given in Ref. 11 for a set of nonlinear renormalization-group equations involving two parameters. This solution describes the crossover between Gaussian and Wilson-Fisher (WF) critical behavior,[4] and is obtained within an ϵ-expansion approximation. The techniques used in the solution of this problem parallel those that are used in Sec. III for a three-parameter crossover problem and, in fact, the three-parameter problem reduces to the two-parameter problem on special surfaces in the Hamiltonian parameter space.

The properties of the global solutions given in Ref. 11 are more fully developed in this section. In particular, we show that not every global solution can match the local, linearized solutions at both of the fixed points. More precisely, a general global solution may be singular on particular trajectories leading from the Gaussian or WF fixed points. If the global solution is to avoid such singularities, then the class of admissible solutions is reduced. In fact, it can be shown that the local, linearized analysis is now justified; the global solution matches the local solutions formed at both the WF and Gaussian points and the asymptotically valid critical behavior is determined by the stabler fixed point.

In this section and in Sec. III we use an approximate renormalization group[16] based on the momentum-independent limit of the differential generator derived by Wegner and Houghton.[9] The use of a differential formulation is far more convenient for the global study of nonlinear equations than an iteration equation such as the Wilson-approximate-recursion formula,[5] since it allows the use of many techniques familiar from the general theory of differential equations.[6,18] A discussion of the Wegner-Houghton–approximate-renormalization group (WHARG) is given in Appendix A.

We consider a Wilson-reduced Hamiltonian density of the form

$$\mathcal{H} = |\nabla s|^2 + r\vec{s}^2 + \tfrac{1}{2}u(\vec{s}^2)^2 + \vec{h}\cdot\vec{s}\ , \tag{2.1}$$

for a continuous spin vector s with n components. The variables r and u are constant interaction parameters and $\vec{h}$ is the magnetic field. As discussed in Ref. 5, the Hamiltonian (2.1) models a short-range interaction between spins on a lattice. The approximation of such a system by a continuum spin allows the renormalization average to be performed more easily. The WHARG equations for the isotropically interacting n-spin system of (2.1) in a lattice of dimension d are

$$\dot{r} = 2r + [u/(1+r)]\,[\tfrac{1}{2}d(n+2)]\ , \tag{2.2a}$$

$$\dot{u} = (4-d)u - [u^2/(1+r)^2]\,[\tfrac{1}{2}d(n+8)]\ . \tag{2.2b}$$

A natural change of variable maps all the fixed points of interest into a finite region of parameter space. We write

$$\begin{aligned} \bar{r} &\equiv r/(1+r)\ , \\ \epsilon y_n &\equiv [u/(1+r)^2]\,[\tfrac{1}{2}d(n+8)]\ . \end{aligned} \tag{2.2c}$$

The WHARG equations in terms of these variables are[23]

$$\dot{\bar{r}} = 2(1-\bar{r})\,[\bar{r} + \epsilon y_n(n+2)/2(n+8)]\ , \tag{2.3a}$$

$$\dot{y}_n = y_n\{\epsilon[1 - 3y_n(n+4)/(n+8)] - 4\bar{r}\}\ . \tag{2.3b}$$

The three fixed points of interest are the "finite Gaussian" point ($\bar{r} = y_n = 0$), the "infinite Gaussian" point ($\bar{r} = 1$, $y_n = 0$), and the Wilson-Fisher point

$[\bar{r} = -\epsilon(n+2)/2(n+8),\ y_n = 1]$. The term "Gaussian" is applied to the first two points since the nonquadratic terms in the effective Hamiltonian are zero. We distinguish between the two Gaussian points by the value of r, which is zero at the finite Gaussian point and infinite at the infinite Gaussian point.

The equations given in (2.3) are already diagonalized about the infinite Gaussian point. If we diagonalize them around the finite Gaussian point, we have the complementary set of equations:

$$\dot{x} = 2x[1 - x - \epsilon(\tfrac{1}{2}\Delta_n) y_n] \tag{2.4a}$$

and

$$\dot{y}_n = y_n[\epsilon(1 - y_n) - 4x] , \tag{2.4b}$$

where

$$x \equiv \bar{r} + y_n \epsilon\, \Delta_n/(2 - \epsilon) , \quad \Delta_n \equiv (n+2)/(n+8) . \tag{2.4c}$$

In passing from (2.3) to (2.4) we have discarded terms proportional to $\epsilon^2 y_n$ and $\epsilon^2 y_n^2$. Thus, (2.3) and (2.4) are formally equivalent to $O(\epsilon)$ (see, however, Appendix B). The advantage of this approximation is that the surface of critical Hamiltonians is now the straight line $x = 0$. We are justified in neglecting the ϵ^2 terms if y_n is always of $O(1)$. Since the original WHARG equations (2.2) are only good to $O(\epsilon)$ [16] this approximation is self-consistent. The restriction on the size of y_n limits us to global solutions for which the renormalization trajectories are bounded in y_n.

With these approximations, (2.3) and (2.4) are of the same form. The three fixed points have been placed at the canonical locations $x = y_n = 0$ (the finite Gaussian point), $x = 0$, $y_n = 1$ (the Wilson-Fisher point) and $x = 1$, $y_n = 0$ (the infinite Gaussian point) (see Fig. 2).

The solution of (2.3) and (2.4) for x and y_n can be conveniently expressed in terms of three functions R, X, and Y_n which are defined through the equations:

$$\dot{R} = 2(1 - \bar{r})R , \tag{2.5a}$$

$$\dot{X} = -2xX , \tag{2.5b}$$

$$\dot{Y}_n = -\epsilon y_n Y_n . \tag{2.5c}$$

Employing (2.5a) and (2.5c) in Eq. (2.3) we have

$$\frac{y_n}{R^2 Y_n^{3(n+4)/(n+8)}} = \text{const}\ e^{-dl} , \tag{2.6a}$$

$$\frac{1 - \bar{r}}{R\, Y_n^{\Delta_n}} = \text{const}\ e^{-2l} . \tag{2.6b}$$

Using (2.5b) and (2.5c) in the complementary equations (2.4) we also have

$$\frac{y_n}{X^2 Y_n} = \text{const}\ e^{el} , \tag{2.6c}$$

$$\frac{x}{X Y_n^{\Delta_n}} = \text{const}\ e^{2l} . \tag{2.6d}$$

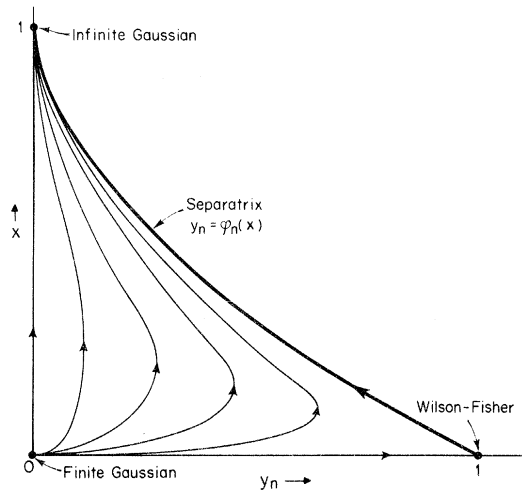

FIG. 2. Solution region for the two-parameter problem of Sec. II is shown. The region includes three fixed points, denoted as the finite Gaussian, infinite Gaussian, and Wilson-Fisher fixed points. The separatrix connecting the Wilson-Fisher and infinite Gaussian fixed points is labeled $y_n = \phi_n(x)$. The line $x = 0$ corresponds to the surface of critical Hamiltonians.

The four expressions on the left-hand sides of (2.6a)–(2.6d) are termed *nonlinear scaling fields*[17] since they have a purely exponential dependence on the renormalization parameter l. Equations (2.6a)–(2.6d) cannot all be independent since there can only be two independent scaling fields. It is easy to see that

$$X = 1 - \bar{r} , \tag{2.7a}$$

$$R = x\, Y_n^{-2\Delta_n} . \tag{2.7b}$$

All that remains is finding the solution of (2.5c) for Y_n. We are interested in the solution that can be written as

$$Y_n = \exp\left(-\epsilon \int_{-\infty}^{l} y_n(l')\, dl'\right) . \tag{2.8}$$

By (2.8) we mean that the value of $Y_n(x, y_n)$ is to be determined by performing the indicated integral along the unique renormalization trajectory that passes through the point (x, y_n). Thus, $Y_n(0, y_n) = 1 - y_n$ by direct integration of (2.5). The separatrix connecting the WF point $x = 0, y_n = 1$ and the infinite Gaussian point $x = 1$, $y_n = 0$ is denoted by $y = \varphi_n(x)$ in Fig. 2. Since the renormalization solution along this trajectory reaches the WF fixed point only in the limit $l \to -\infty$, at each point $y = y_0(l_0)$ on the separatrix, $y > y_0$ for all $l < l_0$. The integral in the exponent of (2.8) diverges and, therefore Y_n has a zero on the separatrix $y = \varphi_n(x)$.

Since we have the exact solution for $x \to 0$, $Y_n = 1 - y_n$, it is easy to show that Y_n can be written as (to first order in ϵ, cf. Appendix B),

$$Y_n = \mathcal{Y}(x, y_n, \Delta_n)$$

$$\equiv (1 - y_n/\varphi_n)\exp(\epsilon x \Delta_n y_n/\varphi_n) , \qquad (2.9a)$$

where the separatrix function $\varphi_n(x)$ shown in Fig. 2 is given by

$$\varphi_n = \Phi(x, \Delta_n)$$

$$\equiv (1-x)^{d/2}\exp[\epsilon x(1-2\Delta_n)/2] . \qquad (2.9b)$$

(The derivation of the functions $\mathcal{Y}$ and Φ is given in Appendix B; they are used with different arguments in Sec. III.) We can now write down the globally valid nonlinear scaling fields appropriate to the Gaussian and WF fixed points. That is, the nonlinear fields that embody the behavior characteristic of the renormalization equations when linearized around the two fixed points. The simplest forms of these fields are given by

$$S_G = x\, Y_n^{-\Delta_n}/X , \qquad (2.10a)$$

$$S_n = x\, y_n^{-\Delta_n}/X^{1-2\Delta_n} . \qquad (2.10b)$$

With these scaling fields, we may describe the global behavior of any function whose renormalization transformation behavior is known. If f is a function that satisfies the renormalization equation $\hat{f} = a_f f$, then f is a GHF of the magnetic field h, and the two scaling fields S_G and S_n,

$$f(\lambda^{a_h} h, \lambda^{a_G} S_G, \lambda^{a_n} S_n) = \lambda^{a_f} f(h, S_G, S_n) . \qquad (2.11)$$

Here the scaling powers a_h, [24] a_G, and a_n are given by

$$a_h = 1 + \tfrac{1}{2}d , \quad a_G = 2 , \quad a_n = 2 - \epsilon\Delta_n . \qquad (2.12)$$

For example the correlation length satisfies (2.11) with $a_f = -1$; the Gibbs potential satisfies (2.11) with $a_f = d$ [see, however, discussion following Eq. (2.15)].

The renormalization-group equations do not determine the form at the GHF (2.11)[11b,25-27] and they also do not determine the scaling fields uniquely. Since any GHF remains a GHF under any transformation of variables which is itself a GHF we may choose new scaling fields which are arbitrary GHF's of S_G and S.

Thus, we can choose scaling fields with any scaling powers. This freedom is reduced by considering the fact that the separatrix connecting the finite Gaussian point and the WF point ($x = 0$, $0 \le y \le 1$) corresponds exactly to the surface of critical Hamiltonians. That is, $x = 0$ if and only if $T = T_c$. Making the usual Taylor-series expansion we assume that $x \approx T - T_c$ for sufficiently small $T - T_c$. If we require that the scaling fields themselves be proportional to x for T near T_c and that they reduce to the natural linearized scaling fields at the Gaussian and WF fixed points, then the most general fields are given by those of (2.12) multiplied by arbitrary functions of the renormalization invariant

$$I \equiv \frac{x^{\epsilon}}{y_n^{\frac{2}{2}}}\, Y_n^{a_n} X^{d} , \qquad (2.13)$$

which do not vanish at $x = 0$. Note that $I = 0$ on the separatrix $y_n = \varphi_n(x)$ as well as when $x = 0$; $I = \infty$ along the pure Gaussian trajectory $y_n = 0$.

This freedom in the choice of nonlinear scaling fields is illusory since we have not specified the GHF's for which the scaling fields are arguments. Any change in the scaling fields induces a corresponding change in the form of the GHF's. We may, therefore, choose the nonlinear scaling fields at our convenience.

Without loss of generality, we will use the scaling fields defined in (2.10). With these nonlinear scaling fields a particularly simple example for the $h = 0$ correlation length is

$$\xi = A(I) S_G^{-1/a_G} + B(I) S_n^{-1/a_n} , \qquad (2.14)$$

where $A(I)$ and $B(I)$ are smoothly varying functions of the renormalization invariant (finite both at $I = 0$ and $I = \infty$). Since both of the scaling fields appear in (2.14) symmetrically, this form has the virtue of reducing to the appropriate local solution as either of the limiting trajectories [$y_n = 0$ or $y_n = \varphi_n(x)$] is approached. For x fixed and $y_n \to 0$, $S_n \to \infty$ and the WF term vanishes. Similarly, as the $y_n = \varphi_n$ separatrix is approached, $S_G \to \infty$ and the Gaussian term $\to 0$. For intermediate values of y_n both singularities compete, giving the expected nonlinear crossover.

A more complicated behavior is exhibited by the Gibbs potential. In addition to the spin dependent terms, an additive constant in the Hamiltonian density contributes to G. Therefore, we can write

$$G(\{p_i\}) + v_0 = e^{-dl} G(\{p_i(l)\}) + e^{-dl} v_0(l) . \qquad (2.15a)$$

The WHARG equation for v_0 can be easily integrated to give

$$G(\{p_i\}) = e^{-dl} G(\{p_i(l)\}) + \frac{dn}{2}\int^{l} dl' \ln[1 + r(l')]\, e^{-dl'} . \qquad (2.15b)$$

As the renormalization average proceeds ($l \to \infty$), information about the Gibbs potential passes from the first term on the right-hand side of (2.15b) to the second term. In some circumstances (in particular, zero magnetization) it may be possible to take the limit $l \to \infty$ and consider only the second term. [3] This method has been utilized by some authors[26,27] to calculate approximate Gibbs potentials in zero ordering field (i.e., $h = 0$). However, for fixed l, the second term contains information that would be unimportant for critical behavior ($x \to 0$). Accordingly, we will in our discussions drop the second term of (2.15b) and deal only with the homo-

geneous term. Thus, when discussing the Gibbs potential and its temperaturelike derivatives, we will confine our attention to $x \to 0$, even though the solutions for the nonlinear scaling fields are valid for all $x < 1$. The difficulty does not arise when studying the derivatives of the Gibbs potential with respect to the ordering field h (such as the magnetization and susceptibility) since the second term in (2.15b) is independent of h and does not contribute. We could therefore phrase our discussion of crossover in terms of these functions; we discuss the Gibbs potential to allow the closest connection between this work and phenomenological discussions of crossover.[20,21] A brief discussion of the limit $l \to \infty$ in the second term is given in Appendix D.

Within this approximation, therefore, the Gibbs potential is a GHF of the ordering field h and the two nonlinear scaling fields S_n and S_G. In general, $G(h, S_G, S_n)$ will generate critical-point exponents that do *not* satisfy exponent inequalities as equalities. This is to be expected on general grounds simply because $G(h, S_G, S_n)$ depends on three scaling fields with three distinct scaling powers. The usual scaling equalities which relate *three* exponents are satisfied because there are only *two* independent scaling powers. An example of a Gibbs potential which is a global solution of the renormalization equations which does not give exponent equalities is

$$G(h, S_G, S_n) = G_G(h, S_G) + G_h(h, S_n) , \tag{2.16}$$

where G_G and G_n are both GHF's. Each piece of the Gibbs potential generates its own singularities with exponents that satisfy exponent equalities. However, since $\gamma_n > \gamma_G$, it is immediate that $\alpha_G > \alpha_n$. The measured exponents would be γ_n and α_G, and therefore $\alpha + 2\beta + \gamma > 2$!

However, a solution of the form given in (2.16) cannot be matched to the expected local solutions near the two limiting trajectories. As $y_n \to 0$, $S_n \to \infty$ and therefore $G_n \to \infty$. Similarly, as the separatrix is approached, $S_G \to \infty$ and $G_G \to \infty$. If we replace G_n by $G_n/(1+I^2)$, the divergence at $y_n = 0$ is removed. However, to remove the divergence on the separatrix we would have to *multiply* G_G by a power of the invariant. Since the invariant is proportional to a power of x [cf. Eq. (2.13)], this weakens the singularities generated by G_G. In fact, it is easy to see that it weakens G_G just enough to ensure that the measured α will be α_n. Thus, if we require that the global solution match the expected local solution on both boundaries a splitting of the Gibbs potential as in (2.16) does not lead to the violation of exponent equalities since the G_G term must be discarded. The critical behavior is determined entirely by the WF point.

To show this in another way, consider the $h = 0$ Gibbs potential. We can write it in two ways,

$$G = (S_G)^{2-\alpha_G} f_G(I) , \tag{2.17a}$$

or

$$G = (S_n)^{2-\alpha_n} f_n(I) . \tag{2.17b}$$

If the asymptotically valid value of α is α_G then $f_G(0)$ is a finite constant. However, $I = 0$ on the $y_n = \varphi_n(x)$ separatrix as well as at $x = 0$; therefore, as the separatrix is approached, f_G is well behaved and, since S_G is singular at $Y_n = 0$, G has a singularity on the separatrix. On the other hand, if the asymptotically valid value of α is α_n, then $f_n(0)$ is finite. It is also finite, therefore, on the separatrix. However, as $y_n \to 0$, the invariant $I \to \infty$. Therefore, the divergence in S_n as $y_n \to 0$ may be cancelled by an appropriate behavior of $f_n(I)$ as $I \to \infty$. An example is given by

$$G = \frac{S_n^{2-\alpha_n} S_G^{2-\alpha_G}}{S_n^{2-\alpha_n} + S_G^{2-\alpha_G}} . \tag{2.18}$$

The form given in (2.18) has the virtue of reducing to the expected local solutions on both bounding trajectories. A form valid for nonzero h which corresponds to (2.18) is

$$G = \frac{G_n(h, S_n) G_G(h, S_G)}{G_n(h, S_n) + G_G(h, S_G)} . \tag{2.19}$$

If $\epsilon < 0$, the argument given above is precisely reversed so that the Gaussian fixed point (which in this case is the stabler fixed point) does dominate the global solution. Thus, in this two-parameter example, the stabler fixed point is always dominant globally.

III. THREE-PARAMETER CROSSOVER; COUPLED ORDER PARAMETERS

In this section we describe the solution to nonlinear renormalization-group equations which involve three parameters. These equations model a system involving two interacting order parameters. There are several realistic systems whose phase diagram may be understood in terms of a model Hamiltonian encompassing the interaction between two (or more) coupled order parameters.

One simple example is provided by the phase diagram of the mixed crystals $Fe_xMn_{1-x}WO_4$ near the quadruple point.[28] In these crystals (which possess a monoclinic wolframite structure), the oxygen ions form a distorted hexagonal-close-packed pattern; half the octahedra spaces are filled with Fe or Mn ions and the other half are filled with W ions. The magnetic structure of $FeWO_4$ (ferberite) is antiparallel in alternate planes ($\uparrow\downarrow$). The magnetic cells for $MnWO_4$ (huebnerite) on the other hand, is quadrupled in the a direction and doubled in the b and c directions ($\uparrow\uparrow\downarrow\downarrow$). Wegner[29] has shown that near the quadruple point of such sub-

stances (defined to be the point at which the paramagnetic phase, Fe-ordered phase, Mn-ordered phase, and a mixed phase are simultaneously in coexistence) the free energy may be represented by a model involving two order parameters with reflection symmetry and a biquadratic coupling term. Depending on the various interaction strengths, the two ordered phases are either separated by a first-order transition or by an additional phase. Other examples of magnetic materials exhibiting similar quadruple point include $Fe(Pd_xPt_{1-x})_3$.[30]

The order-disorder transitions in the ammonium halides provide further examples of systems with coupled order parameters. At sufficiently low temperatures, the NH_4 tetrahedra can have two different types of ordering, parallel and antiparallel, in the cubic structure of the halide ions. The coupling between the two types of ordering is, however, not direct; it is probably mediated by nonordering effects (e.g., magnetoelastic interactions). Model Hamiltonians involving the interaction of the *parallel* ordering and magnetoelastic effects[31] have been used to simulate the order-disorder phenomena of NH_4Cl, leading to a renormalization-group prediction of classical tricritical behavior. A model Hamiltonian which treats magnetoelastic effects and *both* the *parallel* and *antiparallel* ordering is more complicated and will be treated in a separate paper.[32] Because the elastic distortion breaks the reflection symmetry, the model Hamiltonians for NH_4Cl are generally assumed to contain coupling terms different from the simple biquadratic term employed in the mixed-crystal examples discussed above.

There exist a number of systems that *can* be described by model Hamiltonians with biquadratic coupling terms.[33] These include the metamagnets such as $FeCl_2$,[34] systems involving spin-flop transitions,[35] and certain structural phase transitions.[36] It has been shown that such a model provides descriptions not only of classical tricritical points but also "bicritical" and "tetracritical" points[34–36] in the "physical plane." We will demonstrate in the latter part of this section that this model also contains the type of "higher-order critical points" exemplified by the intersection of critical subspaces (as first proposed by Ref. 20) when the phase diagrams are viewed in the multiparameter Hamiltonian space.

A general model Hamiltonian with a biquadratic coupling term has five interaction parameters[33] (as discussed in Appendix A). In this section, we consider the special case of a system in which the two order parameters play precisely equivalent roles. The system considered is a generalization to n-component spins of the anisotropic Hamiltonian discussed in Ref. 4 for $n=1$. The close relationship of this three-parameter system to the two-parameter system solved in Sec. II allows many of the solution methods of the simpler problem to be applied to its generalization in this section.

We consider two internally isotropic n-component spin subsystems, $\vec{s}_1$ and $\vec{s}_2$ which interact through a quartic term,

$$\begin{aligned}\mathcal{H} = &|\nabla \vec{s}_1|^2 + |\nabla \vec{s}_2|^2 + r(\vec{s}_1^{\,2} + \vec{s}_2^{\,2}) \\ &+ \tfrac{1}{2}u[(\vec{s}_1^{\,2})^2 + (\vec{s}_2^{\,2})^2] \\ &+ w\,\vec{s}_1^{\,2}\vec{s}_2^{\,2} + \vec{h}\cdot\vec{s}_1 + \vec{h}\cdot\vec{s}_2 . \end{aligned} \tag{3.1}$$

This Hamiltonian can be viewed as the sum of two n-spin Wilson Hamiltonians of the form given in Eq. (2.1), with a biquadratic interaction term $w\,\vec{s}_1^{\,2}\vec{s}_2^{\,2}$. On the other hand, it can also be considered as the Wilson Hamiltonian for a single $2n$-component spin system $\vec{s} = (\vec{s}_1, \vec{s}_2)$ with an "anisotropy-like" interaction $(w-u)\,\vec{s}_1^{\,2}\vec{s}_2^{\,2}$ (cf. Fig. 3). These two descriptions of the single Hamiltonian (3.1) are reflected in the discussion of the global renormalization properties of (3.1) as will be shown below.

The WHARG equations are given by (see Appendix A for details);

$$\begin{aligned}\dot{\bar{r}} &= 2(1-\bar{r})\left[\bar{r} + \frac{\epsilon}{2}\left(\frac{n+2}{n+8}\,y_n + \frac{n+1}{n+4}\,y_{2n}\right)\right], \\ \dot{y}_n &= y_n\left\{\epsilon\left[1 - 3\left(\frac{n+4}{n+8}\right)y_n - 2y_{2n}\right] - 4\bar{r}\right\}, \\ \dot{y}_{2n} &= y_{2n}\left\{\epsilon\left[1 - 3\left(\frac{n+2}{n+4}\right)y_{2n} - 4\left(\frac{n+2}{n+8}\right)y_n\right] - 4\bar{r}\right\},\end{aligned} \tag{3.2}$$

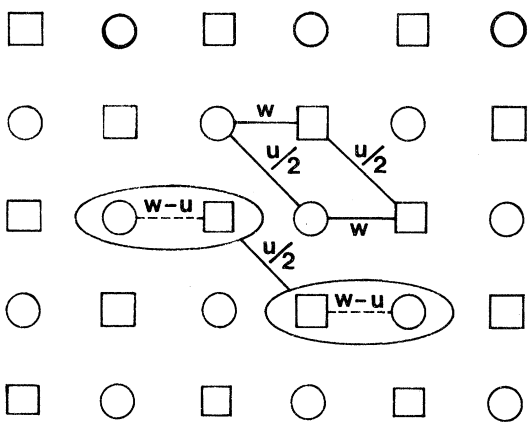

FIG. 3. Diagramatic representation of the Hamiltonian density (3.1). The squares and circles represent the n-component subsystems $\vec{s}_1$ and $\vec{s}_2$, respectively. This system can be regarded as either (a) possessing a biquadratic interaction between the two subsystems, or (b) possessing an anisotropic self-interaction of a single $2n$ component spin $\vec{s} \equiv (\vec{s}_1, \vec{s}_2)$. In case (a), the intrasystem interaction strength is w while the intersystem interaction is $\frac{1}{2}u$. In case (b), the super-spin interaction is $\frac{1}{2}u$ while the anisotropic interaction is $w-u$.

where the variables $\bar{r}$, y_n, and y_{2n} are defined by the relations

$$\bar{r} \equiv r/(1+r),$$
$$\epsilon y_n \equiv \tfrac{1}{2} d(n+8)(u-w)/(1+r)^2, \tag{3.3}$$
$$\epsilon y_{2n} \equiv d(n+4)w/(1+r)^2.$$

The two descriptions mentioned above of the Hamiltonian (3.1) have been incorporated in the choice of the variables y_n and y_{2n} employed here. When $y_n = 0$ the system is equivalent to an isotropic $2n$-component spin system; on the other hand, when $y_{2n} = 0$ the system breaks into two noninteracting n-component spin systems.

Equations (3.2) are already diagonalized around the infinite Gaussian fixed point, $\bar{r} = 1$, $y_n = y_{2n} = 0$. If, as in Sec. II, we diagonalize around the finite Gaussian point $\bar{r} = y_n = y_{2n} = 0$, we obtain the equations

$$\dot{x} = 2x[1 - x - \tfrac{1}{2}\epsilon(\Delta_n y_n + \Delta_{2n} y_{2n})], \tag{3.4a}$$
$$\dot{y}_n = y_n\left[\epsilon\left(1 - y_n - \frac{6}{n+4} y_{2n}\right) - 4x\right], \tag{3.4b}$$
$$\dot{y}_{2n} = y_{2n}[\epsilon(1 - y_{2n} - 2\Delta_n y_n) - 4x], \tag{3.4c}$$

where

$$x \equiv \bar{r} + \epsilon(\Delta_n y_n + \Delta_{2n} y_{2n})/2,$$
$$\Delta_n \equiv (n+2)/(n+8),$$
$$\Delta_{2n} \equiv (n+1)/(n+4).$$

Terms such as $\epsilon^2 y_n$, $\epsilon^2 y_n^2$, $\epsilon^2 y_{2n}$, and $\epsilon^2 y_{2n}^2$ have been neglected. The considerations of Sec. II apply here as well; only global solutions bounded in y_n and y_{2n} are acceptable (see again Appendix B).

The surface of critical Hamiltonians is the plane $x = 0$. There are five fixed points at which x, y_n, and y_{2n} are $O(1)$ [there are other fixed points at which x, y_n, and y_{2n} are $O(1/\epsilon)$; these cannot be subsumed in this perturbation analysis]. Four are located on the plane $x = 0$, and one at $x = 1$: (i) the finite Gaussian point, $y_n = y_{2n} = x = 0$; (ii) the usual n-spin WF point, $y_n = 1$, $y_{2n} = x = 0$; (iii) the usual $2n$-spin WF point, $y_{2n} = 1$, $y_n = x = 0$; (iv) a point of no particular spin (unless $n = 1$) which we will call the z point, $x = 0$, $y_n = y_{n0} \equiv (n-2)(n+8)/(n^2+8)$, $y_{2n} = y_{2n\,0} \equiv (16 - n^2)/(n^2+8)$; (v) the infinite Gaussian point, $x = 1$, $y_n = y_{2n} = 0$. These fixed points and the integral curves and surfaces connecting them form a finite region of the parameter space which is invariant under the action of the renormalization group.

The most relevant eigenvalue [the eigenvalue of the x equation (3.4a)] for the four fixed points in the $x = 0$ plane is given by

$$a_G = 2,$$
$$a_n = 2 - \epsilon\Delta_n,$$
$$a_{2n} = 2 - \epsilon\Delta_{2n}, \tag{3.5}$$
$$a_z = 2 - \epsilon\Delta_z,$$

where $\Delta_z \equiv 3n/(n^2+8)$. Since we identify x with $T - T_c$ for sufficiently small x, the critical-point exponent ν is given by the inverse of the eigenvalues of (3.5); for example, $\nu_G = \frac{1}{2}$.

The existence of the z point shows that the Hamiltonian (3.1) contains a third symmetry relation similar to the two discussed above which are represented by the n-spin and $2n$-spin fixed points.[4] The relative stability of the n-spin, $2n$-spin, and z points depends on the spin dimension n as determined by (3.4). For $n < 2$, the $2n$-spin point is stablest [cf. Fig. 4(a)]. For $2 \leq n \leq 4$, the z point is stablest [Fig. 4(b)]. Finally, for $n > 4$, the n-spin point is stablest [Fig. 4(c)]. The finite Gaussian point is always unstable with respect to all the other $x = 0$ fixed points; all the $x = 0$ fixed points are unstable with respect to the infinite Gaussian fixed point.

From (3.4) we note that the trajectory which joins the finite Gaussian point ($y_n = y_{2n} = 0$) to the z point ($y_n = y_{n0}$, $y_{2n} = y_{2n0}$) is a straight line. Hence, we may supplement (3.4) by

$$\dot{z} = z[\epsilon(1 - y_n - y_{2n}) - 4x], \tag{3.6}$$

where $z \equiv y_n y_{2n0} - y_{2n} y_{n0}$. Note that the z point is $y_n = y_{n0}$, $x = z = 0$. Equation (3.6) is not independent of Eqs. (3.4), but the redundant information expressed in (3.6) will be very helpful in the solution of the original equations (3.4).

Proceeding as in Sec. II, we define functions Y_n, Y_{2n}, X, R, by the equations

$$\dot{R} = 2(1 - \bar{r})R, \tag{3.7a}$$
$$\dot{X} = -2xX, \tag{3.7b}$$
$$\dot{Y}_n = -\epsilon y_n Y_n, \tag{3.7c}$$
$$\dot{Y}_{2n} = -\epsilon y_{2n} Y_{2n}. \tag{3.7d}$$

By inspection of Eqs. (3.2), (3.4), and (3.6), we write down nonlinear scaling fields:

$$\frac{x}{X Y_n^{\Delta_n} Y_{2n}^{\Delta_{2n}}} = \text{const}\, e^{2l}, \tag{3.8a}$$
$$\frac{1 - \bar{r}}{R Y_n^{\Delta_n} Y_{2n}^{\Delta_{2n}}} = \text{const}\, e^{-2l}, \tag{3.8b}$$
$$\frac{y_n}{R^2 Y_{2n}^2 Y_n^{3(n+4)/(n+8)}} = \text{const}\, e^{-dl}, \tag{3.8c}$$
$$\frac{y_{2n}}{R^2 Y_n^{4\Delta_n} Y_{2n}^{3(n+2)/(n+4)}} = \text{const}\, e^{-dl}, \tag{3.8d}$$
$$\frac{y_n}{X^2 Y_{2n}^{6/(n+4)} Y_n} = \text{const}\, e^{\epsilon l}, \tag{3.8e}$$
$$\frac{y_{2n}}{X^2 Y_n^{2\Delta_n} Y_{2n}} = \text{const}\, e^{\epsilon l}, \tag{3.8f}$$

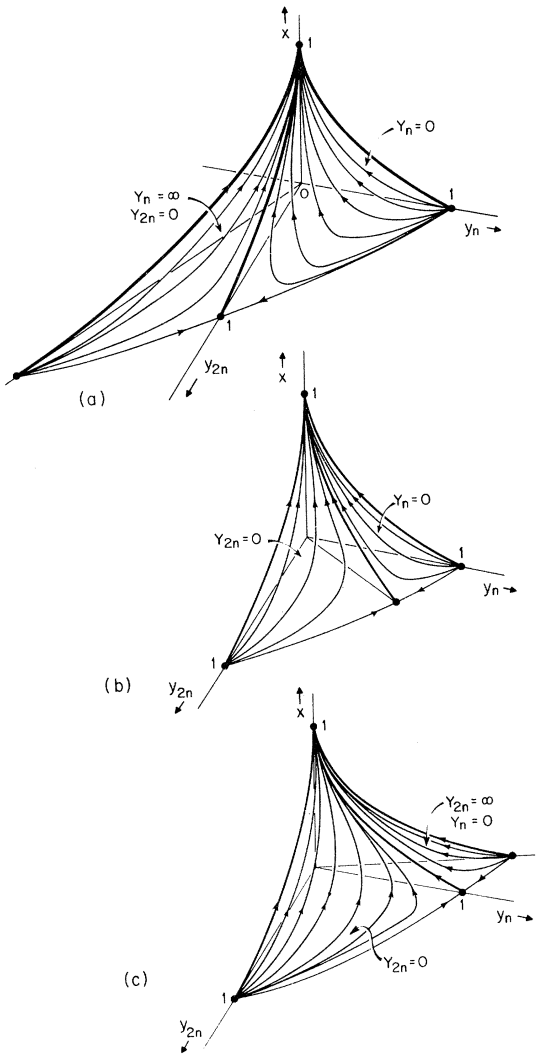

FIG. 4. Solution region for the three-parameter problem of Sec. III is shown for various values of the spin-dimension n. There are five fixed points: the finite Gaussian point, the infinite Gaussian point, the n-spin point at $x=0$, $y_{2n}=0$, $y_n=1$, the $2n$-spin point at $x=0$, $y_n=0$, $y_{2n}=1$, and the "z point" whose location depends on n as indicated in the text. Boundary conditions on the functions Y_n and Y_{2n} are indicated for the separ-surfaces. In (a), the solution region is depicted for $n \leq 2$; in (b), for $2 \leq n \leq 4$; in (c), $n \geq 4$.

$$\frac{z}{X^2 Y_n Y_{2n}} = \text{const}\, e^{\epsilon l}. \tag{3.8g}$$

Matching scaling fields gives $X = (1 - \bar{r})$ and $R = xY^{-2\Delta_n}$, $Y_{2n}^{-2\Delta_{2n}}$ in analogy with (2.7a) and (2.7b). Equations (3.8c)–(3.8g) cannot be independent since z is a linear combination of y_{2n} and y_n. Comparing the solutions for $l \to -\infty$, where Y_n and $Y_{2n} = 1$ gives the following relationship between Y_n and Y_{2n}:

$$y_n y_{2n0} - y_{2n}\, y_{n0} = y_n\, y_{2n0}\, Y_{2n}^{(n-2)/(n+4)} - y_{2n} y_{n0}\, Y_n^{(4-n)/(n+8)}. \tag{3.9}$$

Thus, for $n \neq 2, 4$ the complete solution of the three-parameter crossover problem depends only on the solution of (3.7) for either Y_n or Y_{2n}. Unfortunately (except for $n = 0$ and $n = \infty$, see Appendix C), we have not been able to derive the form of Y_n or Y_{2n} even in an ϵ expansion. The essential difficulty is in the $x = 0$ plane, where there is no small parameter and the nonlinear equations must be solved exactly. However, a great deal can be learned about the solutions by comparison with the two-parameter solutions of Sec. II on appropriate two-dimensional surfaces. Whenever y_n, y_{2n}, or z vanishes, the problem reduces to a two-parameter problem. We have, therefore, the following partial results:

$$Y_n(x, y_n, y_{2n} = 0) = \mathcal{Y}(x, y_n, \Delta_n), \tag{3.10a}$$

$$Y_{2n}(x, y_n = 0, y_{2n}) = \mathcal{Y}(x, y_{2n}, \Delta_{2n}), \tag{3.10b}$$

$$Y_n(x, y_n, y_{2n})\big|_{z=0} = [\mathcal{Y}(x, y_n/y_{n0}, \Delta_z)]^{y_{n0}}, \tag{3.10c}$$

$$Y_{2n}(x, y_n, y_{2n})\big|_{z=0} = [\mathcal{Y}(x, y_n/y_{n0}, \Delta_z)]^{y_{2n0}}. \tag{3.10d}$$

To discuss the boundary conditions that apply to Y_n and Y_{2n} we must generalize the notion of separatrix. A fixed point at which not all the eigenvalues of the linearized equations are of the same sign is a saddle point in the appropriate space. The family of trajectories leaving a saddle point in a two-dimensional space is a one-dimensional line (a separatrix). In higher-dimensional spaces, this family of trajectories may be of higher dimension, and we will call the corresponding surface a "separ-surface". On the separ-surface leaving the n-spin point, there is a boundary condition on Y_n; on the separ-surface leaving the $2n$-spin point, there is a boundary condition on Y_{2n}; on the separ-surface leaving the z point, boundary conditions on both Y_n and Y_{2n} apply. As in the discussion following (2.8), each Y on which a condition is imposed must be identically zero or infinite on the appropriate surface. These boundary values are shown in Fig. 4 for the three ranges of n, $n \leq 2$, $2 \leq n \leq 4$, and $n \geq 4$. The general character of Y_n and Y_{2n} is established, even though the solutions cannot be given explicitly. We will proceed as if Y_n and Y_{2n} were known (for $n = 0$ and $n = \infty$, Y_n and Y_{2n} are known, see Appendix C).

Since there are *three* parameters (x, y_n, y_{2n}) there are *two* independent renormalization invariants. It will be convenient to write down several nonindependent invariants for compactness in later discussion. There are three ways of combining (3.8e)–(3.8g):

$$I_{y_n y_{2n}} \equiv \frac{y_n Y_{2n}^{(n-2)/(n+4)}}{y_{2n} Y_n^{(4-n)/(n+8)}}, \tag{3.11a}$$

$$I_{z y_n} \equiv \frac{z}{y_n Y_{2n}^{(n-2)/(n+4)}}, \tag{3.11b}$$

$$I_{zy_{2n}} \equiv \frac{z}{y_{2n} Y_n^{(4-n)/(n+8)}} , \tag{3.11c}$$

which are connected through (3.9). Combining (3.8a) with (3.8e)–(3.8g) we have three invariants involving x explicitly

$$I_{xy_{2n}} \equiv \frac{x^{\epsilon} X^{d} Y_{2n}^{a_{2n}} Y_n^{a_n - 2(4-n)/(n+8)}}{y_{2n}^{2}} , \tag{3.12a}$$

$$I_{xy_n} \equiv \frac{x^{\epsilon} X^{d} Y_n^{a_n} Y_{2n}^{a_{2n} - 2(n-2)/(n+4)}}{y_n^{2}} , \tag{3.12b}$$

$$I_{xz} \equiv \frac{x^{\epsilon} X^{d} Y_n^{a_n} Y_{2n}^{a_{2n}}}{z^{2}} . \tag{3.12c}$$

Finally we combine (3.11) and (3.12) to give an invariant which does not contain either Y_{2n} or Y_n,

$$I_0 \equiv \frac{x^{\epsilon} X^{d} z^{a_z (n^2+8)/(4-n)(n-2)}}{y_n^{a_n (n+8)/(4-n)} y_{2n}^{a_{2n}(n+4)/(n-2)}} . \tag{3.13a}$$

The invariant I_0 of (3.13) is useful because it distinguishes between different trajectories on the curved separ-surfaces where either Y_n or Y_{2n} is identically zero or infinite. The invariants (3.11) and (3.12), which involve Y_n and Y_{2n} explicitly, are constant on these separ-surfaces, while (3.13) varies. For $n=2$ and $n=4$ it is impossible to form an invariant that contains neither Y_n nor Y_{2n}. However, in these cases it is not necessary to eliminate both of the y's since only one of them has singular behavior on the separ-surface. For $n=2$, $Y_n=0$ on the separ-surface and we can choose the invariant

$$I_2 \equiv \frac{x^{\epsilon} X^{d} Y_{2n}^{a_{2n}}}{y_{2n}^{a_n (n+8)/(4-n)} z^{2 - a_n (n+8)/(4-n)}} , \tag{3.13b}$$

in this case. For $n=4$, $Y_{2n}=0$ on the separ-surface, and the invariant

$$I_4 \equiv \frac{x^{\epsilon} X^{d} Y_n^{a_n}}{y_n^{a_{2n}(n+4)/(n-2)} z^{2 - a_{2n}(n+4)/(n-2)}} , \tag{3.13c}$$

distinguishes trajectories. For the remainder of this work it is simplest to assume that $n \neq 2, 4$, although the analysis carries over to those cases, via slightly modified arguments.

We may choose the nonlinear scaling fields to be given by

$$S_G \equiv \frac{x}{X Y_n^{\Delta_n} Y_{2n}^{\Delta_{2n}}} , \tag{3.14a}$$

$$S_z \equiv \frac{x}{X^{1-2\Delta_z} y_{2n}^{\Delta_z}} \left(\frac{y_n}{y_{2n}} \right)^{(n^2-4)/(n^2+8)} , \tag{3.14b}$$

$$S_n \equiv \frac{x}{X^{1-2\Delta_n} y_n^{\Delta_n} Y_{2n}^{(n-1)/(n+8)}} , \tag{3.14c}$$

$$S_{2n} \equiv \frac{x Y^{(n^2-4)/(n+4)(n+8)}}{X^{1-2\Delta_{2n}} y_{2n}^{\Delta_{2n}}} . \tag{3.14d}$$

The choice of the scaling fields is (as in Sec. II) not unique, but it is a convenient choice for the case $2 \leq n \leq 4$ which is discussed in detail below as a concrete example. In this case, both y_n and y_{2n} are both positive throughout the solution region.

The invariant I_0 also contains information regarding relative stability. It is zero in the $x=0$ plane and also in that plane which contains the stablest fixed point. For example, in the case $2 \leq n \leq 4$, the z point is stablest and $I_0=0$ whenever $z=0$. The invariant I_0 is infinite in those planes which contain the relatively unstable points. For $2 \leq n \leq 4$, $I_0=\infty$ when $y_n=0$ or $y_{2n}=0$.

As in Sec. II we can show that global solutions which do not generate the exponents of the stablest fixed point have extraneous singularities. For concreteness, we will consider $2 \leq n \leq 4$; a similar analysis can be made for any value of n. If we were to suppose that the asymptotically valid value of α were that given by the n-spin fixed point, rather than that of the z point, it would be appropriate to write the Gibbs free energy as

$$G = S_n^{d/a_n} F(\text{invariants}) , \tag{3.15}$$

since $d/a_n = 2 - \alpha_n$. It is presumed in writing the free energy in the form (3.15) that F(invariants) is well behaved at $x=0$ plane (where we definitely expect n-spin point behavior). The amplitude of the free-energy singularity is given by

$$\lim_{x \to 0} (G/x^{d/a_n}) = y_n^{-\Delta_n d/a_n} Y_{2n}^{-(d/a_n)(n-1)/(n+8)} \times F(\text{invariants})\big|_{x=0} . \tag{3.16}$$

If we consider a path in the $x=0$ plane that is a renormalization trajectory (cf. Fig. 5) (3.11) is again constant. The function F is therefore constant and we see that the amplitude of the free-energy singularity diverges as the stablest fixed point is approached along any renormalization trajectory ($Y_{2n} \to 0$). The singularity can be removed along any particular trajectory or finite number of trajectories but it cannot be removed everywhere in the $x=0$ plane. Similar difficulties are encountered if we assume that the $2n$-spin point dominates the critical behavior of G.

If, however, we assume that the z point dominates the exponents everywhere (except in the $y_n=0$ and $y_{2n}=0$ planes), then we can avoid singularities. We may define a scaling field

$$S_z' \equiv \frac{S_z (1 + S_z^{-1} S_{2n}^{a_z/a_{2n}})}{1 + S_z S_n^{-a_z/a_n}} . \tag{3.17}$$

S_z' reduces to S_z as $x \to 0$, for $y_n \neq 0$, $y_{2n} \neq 0$; as

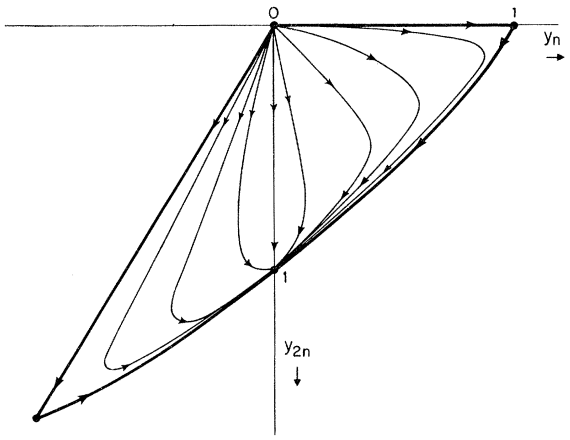

FIG. 5. Solution trajectories in the $x=0$ plane are shown. The renormalization trajectories sweep toward the separatrices joining the relatively unstable fixed points to the stablest fixed points before moving to the stablest fixed point. This behavior, although illustrated for $n \approx 1$ holds for all n.

$y_{2n} \to 0$, $x \neq 0$, $S_z' \to S_n^{a_z/a_n}$; as $y_n \to 0$, $x \neq 0, S_z' \to S_{2n}^{a_z/a_{2n}}$.

Thus a global solution for the Gibbs potential which has z-point-like exponents (except in the special symmetry planes $y_n = 0$ and $y_{2n} = 0$) could be given by

$$G = \frac{G_G(h, S_G) G_z(h, S_z')}{G_G(h, S_G) + G_z(h, S_z')} \quad . \tag{3.18}$$

The linear local analysis is again supported.

We may describe the crossover involved in the scaling field (3.17) and the Gibbs potential (3.18) by noting that if $x \to 0$ and y_n and y_{2n} fixed and nonzero, then only the magnetic field h and S_z "scale." Near the special planes $y_n = 0$ or $y_{2n} = 0$, an additional field is important (S_{2n} or S_n, respectively) and "scales."

It would be more customary to describe this in an alternate manner. Away from the special symmetry planes, S_z scales, and y_n and y_{2n} are truly irrelevant variables. Thus, at each point of the critical surface, $x = 0$, we may consider x to be a scaling variable with y_n and y_{2n} being irrelevant variables which have only an insigificant effect on the thermodynamic functions. Near the junction of the plane $x = 0$ and one of the symmetry planes (for example, $y_{2n} = 0$), y_{2n} is clearly important. Phenomenologically, we might expect x and y_{2n} to scale, while y_n remains irrelevant. To see that this is the case, note that S_z can be written as

$$S_z = S_n \left(\frac{y_{2n}}{Y_{2n} X^2 Y_n^{2\Delta n}} \right)^{\Delta_n - \Delta_z} I_{y_n y_{2n}}^{\Delta_n + (n^2-4)/(n^2+8)} \quad . \tag{3.19}$$

Thus, the scaling invariant combination $S_z S_n^{-a_z/a_n}$ in (3.17) can be written (apart from the irrelevant quantities y_n, Y_n, Y_{2n}, and X) as

$$S_z S_n^{-a_z/a_n} \approx \left(\frac{x^{\epsilon(4-n)/(n+8)}}{y_{2n}^{a_n}} \right)^{[(n+4)(n-1)/(n^2+8)]/a_n} \tag{3.20}$$

This is exactly the expected scaling invariant (which determines the "crossover cones"[21]) involving x and y_{2n} providing that we identify the scaling power of y_{2n} as $a_y \equiv \epsilon(4-n)/(n+8)$ and the scaling power of x as $a_x \equiv a_n$. An examination of (3.4) shows that these are the scaling powers that are obtained by linearizing around the n-spin fixed point. The nonlinear scaling field S_z' embodies the behavior of both fixed points in such a way as to generate the "double-power" scaling laws used to describe crossover in anisotropic systems[37] and in higher-order systems.[20,21] For example, the zero-field Gibbs potential is given by

$$G = \left(\frac{x}{y_n^{(n+4)(n-1)/(n^2+8)} + (x^{a_y/a_x})^{(n+4)(n-1)/(n^2+8)}} \right)^{2-\alpha_z} \tag{3.21}$$

We note that a_y/a_x is a "crossover exponent" and $(2-\alpha_z)(n+4)(n-1)/(n^2+8)$ is an "amplitude exponent."[38] In Eq. (3.21) the dependence on the unimportant quantities y_n, X, Y_n, and Y_{2n} has been ignored. Precisely analogous behavior is found near the junction of the $x = 0$ and $y_n = 0$ planes, where the variables y_n and x appear in a scaling combination with scaling powers derived by linearization around the $2n$-spin fixed point.

The analogy between the nonlinear effect incorporated in the nonlinear scaling fields and the crossover effects in systems with critical points of higher order can be extended. Figure 6(a) shows a three-dimensional section of the four-dimensional phase diagram of the "Ising metamagnet"[39] with interactions J in the plane and $\mathcal{R}J$ between planes ($\mathcal{R} < 0$); the staggered field H' is zero in Fig. 6(a). By varying the strength of the interaction parameter $\mathcal{R}$, the line of ordinary critical points of a simple metamagnet sweeps out a surface of critical points. The tricritical points which marked the transition between the second-order critical behavior and the first-order transition become lines of tricritical points. At $\mathcal{R} = 0$, the system reduces to a set of two-dimensional ferromagnetic systems, and the tricritical lines meet on the T axis at the $d = 2$ critical temperature. As $\mathcal{R} \to \infty$, the system becomes a one-dimensional antiferromagnet; there is no $T \neq 0$ phase transition. The surface of ordinary critical points shrinks and the tricritical lines meet at $T = 0$. The classification system of Ref. 20 terms the surface of ordinary critical points a surface of critical points of order three. The special point at $\mathcal{R} = 0$ is a critical point of order four. This notion of order corresponds both to the number of phases which are simultaneously critical at the critical point and to the number of variables

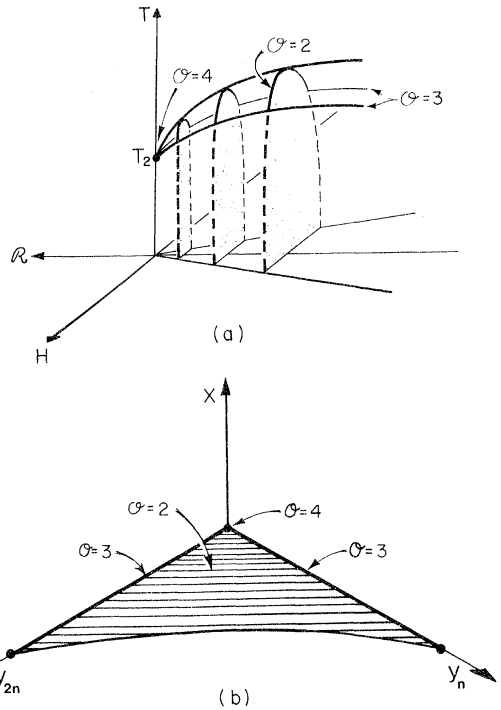

FIG. 6. Comparison of the phase diagram of an Ising metamagnet and the three-parameter crossover problem of Sec. III. (a) The phase diagram of the metamagnet is shown in zero staggered field. A coexistence volume is capped by a surface of ordinary critical points ($\mathcal{O}=2$). This surface is bounded by two lines of tricritical points ($\mathcal{O}=3$). The tricritical lines intersect at the $\mathcal{O}=4$ point $H=0$, $\mathcal{R}=0$, $T=T_2$, the two-dimensional Ising critical temperature. (b) The solution region for the three-parameter problem for $2 \leq n \leq 4$ is shown. The finite Gaussian point corresponds to the point of order-4; the lines $y_n=0$ and $y_{2n}=0$ correspond to the $\mathcal{O}=3$ lines; the portion of the $x=0$ plane bounded by the "tricritical" lines and the separatrices joining the $2n$-spin point and the n-spin point to the z point correspond to the surface of ordinary critical points ($\mathcal{O}=2$).

which "scale" at that point. Thus, Ref. 20 proposes that on the surface of order-two points, two variables scale; that is, the singular part of the Gibbs potential is a GHF of two of the variables (while the dependence on the remaining variables is smooth and non singular). At a point on one of the lines of order-three points, three variables scale. Finally, at the fourth-order point, all four variables scale.

To compare this to the coupled-order parameter system described in this section, we note that near the plane $x=0$ (but not near the lines $y_n=0$ or $y_{2n}=0$) only the variable x and the magnetic field h scale; that is, the leading singular behavior of any function derived from the Gibbs potential (3.18) depends only slightly on the variables y_n and y_{2n} (and the nonlinear functions X, Y_n, and Y_{2n}). However, near the junction of the $x=0$ and $y_{2n}=0$ planes, y_{2n} does not appear merely as a smoothly varying parameter in the amplitudes of the thermodynamic function, but rather in an important, characteristically "double-power" scaling manner. Similarly, near the Gaussian fixed point, all the variables x, y_n, and y_{2n} appear in a "triple-power" scaling formula. We may tentatively relabel the Hamiltonian parameter space as in Fig. 6(b). The $x=0$ plane is a surface of order-2 critical points; the lines $y_n=0$ and $y_{2n}=0$ are "tricritical" lines or lines of order 3; finally, the Gaussian fixed point is a point of order 4.

The order-3 lines of the nonlinear crossover problem [Fig. 6(b)] do not merge smoothly at the order-4 point while those of the metamagnet do [Fig. 6(a)]. This is to be expected since the scaling powers of y_n and y_{2n} are equal (to ϵ) at the order-4 point. A similar situation occurs in the metamagnet. In this case, it is the scaling powers of the direct and staggered fields which are equal; the coexistence volumes do not merge smoothly at the order-4 point.

We also remark that the attainment of a point of order 4 in a space of four dimensions is achievable in both the metamagnet and the renormalization-group examples only due to their highly symmetric nature. In general, a space of dimension 6 is required to observe a point at which four phases are simultaneously critical.[40] Indeed, a less symmetric version of (3.1) discussed in Appendix A is expressed in a six-parameter space.

IV. DISCUSSION

In Secs. II and III we were able to explicitly carry out the solution of the nonlinear WHARG equations. Summarizing the features of these solutions which we believe are of a general nature, we have shown that:

(i) Global solutions *can* be found in a limited but finite region of the Hamiltonian parameter space. This region includes the competing fixed points and is bounded by separ-surfaces emanating from the fixed points.

(ii) Global solutions which do not yield the critical-point exponents derived from the local, linearized analysis of the stablest of the fixed points (on the critical surface) are singular on the separatrix leaving that fixed point.

The properties of the "regular" global solutions are closely analogous to those properties proposed in a phenomenological manner for crossover behavior between various critical points of higher order.[30] For the former, it is the nonlinear character of the scaling equation and scaling fields that embodies several types of ordering and critical behavior. For the latter, the simultaneous validity to lowest

order of several linear scaling equations with linear scaling fields is presumed.[21] In both cases, the borders of a region where m variables scale is a region where $(m+1)$ variables scale.

As shown in Sec. III for a three-parameter example, there is some surface (of dimension greater than 2, in general) of order-2 critical points. The exponents everywhere on this surface are determined by linearization about the stablest fixed point, which is located somewhere on that surface. This surface will be partially bordered by special symmetry "planes" on which the renormalization-group equations involve fewer parameters. In these special surfaces, another fixed point determines the critical behavior. Near the junction of the symmetry plane and the critical surface the two fixed points are in sharp competition. On the border of the order-two critical points, three variables will scale in a characteristically "double-power" law scaling form [cf. Eq. (3.21)]. If the special symmetry planes associated with each such "tricritical" line intersect (as in Sec. III), more parameters are removed from the renormalization-group equations, a new fixed point controls the exponents, and characteristic "triple-power" scaling behavior results. As in the phenomenological studies, this process can be continued indefinitely. As more and more symmetry restrictions are placed on the Hamiltonian, fixed points of weaker and weaker stability determine the critical-point exponents. Since in the immediate neighborhood of a truly unstable fixed point there must be regions controlled by more stable fixed points, the crossover effects get more and more complicated as the order of the critical point increases. *All of these crossover effects are automatically incorporated into the nonlinear scaling fields.*

The above discussion gives reassuring support both to the usual local linearized fixed point analysis and also to the phenomenological descriptions of crossover. However, there are further questions about the behavior of real systems modeled by renormalization-group equations.

The solutions developed in Secs. II and III are only valid in a specified region of the parameter space. For the two-parameter problem (at the critical temperature), $0 \leq y_n \leq 1$. For the three-parameter problem, y_n and y_{2n} are confined to the region enclosed by the lines $y_{2n}=0$, $y_n=0$ and the two separatrices joining the n-spin and $2n$-spin points to the stabler z point (for $2 \leq n \leq 4$). Portions of these boundaries can be understood on a physically intuitive level. In the two-parameter case the restriction $y_n \geq 0$ is necessary for thermodynamic stability. The parameter y_n is proportional to the coefficient fo the quadratric term in the Hamiltonian density; since the quadratic term is the term of highest degree in the Hamiltonian its coefficient must be positive. In the three-parameter problem the stability requirements are $u>0$ and $w>-u$. These are not necessarily the "tricritical" lines (for $2 \leq n \leq 4$, $y_n=0$ and $y_{2n}=0$ are the order-three lines; these restrictions are the lines $u=0$ and $w=u$). However, if these lines do mark the boundary of a region of *first-order transition*, as in the metamagnetic analog, they do form natural borders for the scaling behavior. However, the portion of the boundary formed by the restriction $y_n<1$ in the two-parameter case and by the separatrices in the three-parameter case have no such intuitive explanation.

Within the confines of the region of scaling behavior, the renormalization trajectories are bounded. We may imagine changing the values of the parameters y_n and y_{2n} until the system Hamiltonian lies outside the solution region. In this case, the parameters y_n and y_{2n}, which (except near the "tricritical" lines) did not significantly affect the critical behavior, have unbounded renormalization trajectories. The approximations employed when $\epsilon y_n \ll 1$ and $\epsilon y_{2n} \ll 1$ are no longer valid. The problem immediately becomes far more complicated and it is no longer possible to discuss the renormalization-group solutions within a simple perturbative scheme. Therefore, we can only speculate that the nonlinear solutions will involve many new fixed points and qualitatively different behavior.

Thus, although we have seen that thermodynamic stability requirements and possible first-order transitions may account for some portion of the bounding surface of the solution region, the boundary formed by the separ-surfaces is more complex and possibly marks a transition to vastly altered behavior. However, *it is precisely on this surface that regularity conditions were imposed on the global solutions*. This is not an obvious step.

The requirement of regularity everywhere on the bounding surface corresponds to the notion that we can smoothly move the Hamiltonian (in particular, the critical Hamiltonian) to the "singular separatrix"[41] (in particular, the stablest fixed point). It might seem reasonable to require that such a procedure have a finite limit (as in Sec. III). This resolution has several difficulties, among which are the following:

(i) The bounding separ-surfaces divide the region where even a general global solution is well behaved from a region in which (as argued above) the global behavior may be radically different. It is perhaps over optimistic to ascribe to the boundary between two such regions all the properties of one or the other of the regions.

(ii) It may not be possible by any application of an external field to move the parameters arbitrarily close to the boundary and the singular separatrix. This difficulty is unlikely in the particular cases

treated in Secs. II and III since the fixed-point values of the parameters are small, but it is not an impossible occurence. For example, we showed in Ref. 11 that temperature trajectories for the two-parameter problem usually terminate at a finite value of x (with $x<1$) instead of reaching the infinite Gaussian fixed point $x=1$.

A more cogent example is provided by the five-parameter crossover model described in Appendix A. We consider a system with $n+m$ spin components which decomposes into an n-spin system and an m-spin system.[13] For $n=1$ and $m=2$, this is a description of a Heisenberg ferromagnet with a single axis of anisotropy. The stablest fixed point is either Ising-like (one "easy axis") or XY-like (one "hard axis"). However, these fixed points consist of the XY-subsystem parameters taking on their usual WF fixed-point values, while the Ising subsystem is at its infinite Gaussian fixed point. Loosely, one subsystem is at its critical temperature while the second subsystem is at "infinite" temperature. No physical system with finite Hamiltonian parameters can be *at* such a fixed point; at best, it represents a limit of realizable systems. Therefore, we might expect that any singularity in a thermodynamic function would not manifest itself in the physical space.

Thus, in some cases (not necessarily those treated in Secs. II and III), the requirement of regularity everywhere on the bounding surface may be too stringent. If this requirement is relaxed, nonscaling behavior results. For example, in the three-parameter problem discussed in Sec. III, we might allow free competition among the n-spin fixed point, the $2n$-spin fixed point, and the z point, regardless of stability. These solutions, as shown in Sec. III, *will be singular* on the separatrix leaving the stablest of the $x=0$ fixed points. If this singularity is tolerated, the asymptotically valid critical-point exponents will be determined by the fixed point which contributes the largest singularity to the thermodynamic quantity considered. Therefore, ν and γ will be determined by the fixed point which gives the smallest eigenvalue in the renormalization equation for x (3.4a); on the other hand α and β will be determined by the fixed point which yields the largest such eigenvalue. Since the competition is among three nontrivial fixed points, we need only consider the relative sizes of a_n, a_{2n}, and a_z.

If, for concreteness, we confine our attention to $n>0$, we have $\Delta \geq \Delta_n$ and $\Delta_{2n} \geq \Delta_z$. For $1 \leq n \leq 4$, we also have $\Delta_z \geq \Delta_n$. Thus for $n>0$ the correlation length exponent ν and γ $(=2\nu)$ will be that of the $2n$-spin point, while α and β are given by their n-spin point values for $1 \leq n \leq 4$, and the z point otherwise. Instead of equality in the relationship $\alpha+2\beta+\gamma=2$, we have that

$$\alpha+2\beta+\gamma=2+\frac{\epsilon 3n}{2(n+4)(n+8)}, \quad 1 \leq n \leq 4 \tag{4.1a}$$

$$\alpha+2\beta+\gamma=2+\frac{\epsilon(n-2)^2(n+2)}{2(n+4)(n^2+8)}, \quad n<1, \ n>4. \tag{4.1b}$$

If the finite Gaussian fixed point is allowed to compete as well, ν and γ are as above; however $\alpha=\frac{1}{2}\epsilon$ and $\beta=\frac{1}{2}-\frac{1}{4}\epsilon$, independent of n. The Rushbrooke inequality is

$$\alpha+2\beta+\gamma=2+\epsilon(n+1)/2(n+4). \tag{4.1c}$$

The crossover effects embodied in a general global solution are similarly more complicated than those of the regular solutions exemplified in (3.18). It is reasonable to impose the condition that, in the symmetry planes $y_n=0$, $y_{2n}=0$, and $z=0$ (where the renormalization equations involve only two parameters) any global solution should reduce smoothly to a two-parameter solution (except at the intersection of the separ-surfaces and the symmetry planes). This can always be done by using the invariants (3.11)–(3.13). The crossover near such a symmetry plane is between one set of exponents determined by the free competition of the fixed points in the *three-parameter space* and another set of exponents derived from a similar competition in a two-parameter space. Neither set of exponents need satisfy scaling equalities. This is in contrast to the situation described in Sec. III; in that case, both sets of exponents separately satisfy scaling equalities.

Even though they do not satisfy exponent equalities, general global solutions share other properties with the regular solutions (3.18). The eigenvalue of the magnetic field a_h is larger than a_n, a_{2n}, or a_z. Differentiation with respect to h increases the singularity of the Gibbs potential more rapidly than differentiation with respect to the temperature T. Therefore, we may still describe h as a "strong" direction and T as a "weak" direction in the sense of Griffiths and Wheeler.[42] Elsewhere,[22] we have considered systems which have this property that derivatives taken in different directions have different well-defined relative strengths. We term such systems "critically ordered" and have shown that under certain conditions, the geometric postulates of Griffiths and Wheeler are satisfied for these nonscaling systems. Using this terminology, we can restate the distinction between the general and regular solutions for the thermodynamic functions. The former represents a system that is only critically ordered, while the latter has a true asymptotic scaling form.[43]

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APPENDIX A: THE WEGNER-HOUGHTON-APPROXIMATE-RENORMALIZATION GROUP

Wegner and Houghton[9] have introduced an exact differential generator for the renormalization group as applied to continuum spin Hamiltonian densities. It takes the form of a highly nonlinear functional integrodifferential equation. The Hamiltonian density $\mathcal{H}(\vec{s})$ must be written as a functional of the Fourier transform of the spin density $\vec{s}(\vec{x})$. Thus it is conventional to write an expansion for the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H}=\sum_{j=0}^{\infty}\sum_{k_1\cdots k_j}\sum_{\vec{\alpha}} v_j^{\vec{\alpha}}(\vec{k}_1,\ldots,\vec{k}_j) \times s_{\alpha_1}(\vec{k}_1)\cdots s_{\alpha_j}(\vec{k}_j)\,, \tag{A1}$$

where $\vec{s}(\vec{k})$ is the Fourier transform of $\vec{s}(\vec{x})$ and α is a j-component index, $\vec{\alpha}=(\alpha_1,\ldots,\alpha_j)$. The coefficient functions $v_j^{\vec{\alpha}}(k)$ are also to be considered as functions of the renormalization parameter, l. The renormalization-group equations become nonlinear integrodifferential equations coupling the $v_j^{\vec{\alpha}}(k)$. Equations of this form are nearly intractable. Following the lead of Wilson's approximate recursion formula, we hope that a certain "momentum-independent" or "zero-momentum" limit of the full renormalization-group structure will preserve the basic content of the renormalization-group approach. We force the momentum-dependent coupling constants to be of the form

$$v_j^{\vec{\alpha}}=v_j^{\vec{\alpha}}(0)+\delta_{j,2}\delta_{\alpha_1,\alpha_2}k^2\,. \tag{A2}$$

This is equivalent to choosing a Wilson-reduced Hamiltonian density of the form

$$\mathcal{H}(s)=\left|\vec{\nabla}\vec{s}(\vec{x})\right|^2+H(\vec{s}(\vec{x}))\,, \tag{A3}$$

where

$$H(\vec{s})=\sum_{j=0}^{\infty}\sum_{\vec{\alpha}} v_j^{\vec{\alpha}}\, s_{\alpha_1}\cdots s_{\alpha_j}\,.$$

Hamiltonian densities such as (A3) are, of course, not renormalization-group invariant. Thus, terms which arise from the exact group equations must be discarded if they do not retain the form (A3). This requires, for example, that the critical-point exponent η be set equal to zero since it cannot be determined from the resulting equations.

For the momentum-independent part of the Hamiltonian density (A3), $H(\vec{s})$ we find that the WHARG equation is given by

$$\dot{H}=dH+\tfrac{1}{2}(2-d)\vec{s}\cdot\vec{\nabla}_s H+\tfrac{1}{2}d\ln\det(1+\tfrac{1}{2}\hat{H})\,, \tag{A4}$$

where $\hat{H}$ is the matrix of second partial derivatives,

$$\hat{H}_{ij}\equiv\frac{\partial^2 H}{\partial s_i\partial s_j}\,, \tag{A5}$$

and d is the lattice dimension. Expanding the function $\ln\det$, (A4) can also be written as

$$\dot{H}=dH+\tfrac{1}{2}(2-d)\vec{s}\cdot\vec{\nabla}_s H +\tfrac{1}{2}d\left(\tfrac{1}{2}\,\mathrm{Tr}\hat{H}-\tfrac{1}{8}\,\mathrm{Tr}\hat{H}^2+\tfrac{1}{24}\,\mathrm{Tr}\hat{H}^3-\cdots\right). \tag{A6}$$

This however, may not be the best form of the expansion (see below).

The solutions of the linearized form of (A6), in which the traces of H^2, H^3 and so on are discarded, are the Gaussian fixed-point eigenfunctions. They are given by products of generalized Laguerre polynomials and harmonic polynomials just as in the Wilson approximate recursion formulas[5,44]

$$Q_{m,k}=L_m^{k-1+n/2}\left(\frac{d-2}{d}\,s^2\right)P_k(\vec{s})\,, \tag{A7}$$

where $P_k(\vec{s})$ is a harmonic polynomial of degree k, and n the spin dimension. For a system in dimension n, there are $(2k+n-2)(k+n-3)!/k!(n-2)!$ harmonic polynomials of degree k.[45] All of these polynomials, moreover, are degenerate with respect to the renormalization group since the eigenvalue of the linearized renormalization-group equation depends only on m and k:

$$\dot{Q}_{m,k}=dQ_{m,k}+\tfrac{1}{2}(2-d)\vec{s}\cdot\vec{\nabla}Q_{m,k}+\tfrac{1}{4}d\nabla^2 Q_{m,k} =[d+(2-d)(m+k/2)]\,Q_{m,k}\,. \tag{A8}$$

Elsewhere,[16] we have given a study of the fixed points determined by perturbation from the Gaussian fixed points for the isotropic cases (m arbitrary, $k=0$) and for the special case $n=1$. These problems were considered only from the viewpoint of location of, and linearization about, particular fixed points.

It is easy to check that the solutions of the linearized WHARG equations are solutions of the full Wegner-Houghton–momentum-dependent equations when similarly linearized [for $\eta=0$; if we wish to insert an *a priori* determined η the $(d-2)$ in the argument of the Laguerre polynomial changes to $d-2+\eta$)[16b]]. Thus if the Hamiltonian is "small," the error in using the WHARG equations is of second order of smallness. As a parenthetical remark we note that the class of solutions to the linearized momentum-dependent equations is very large. Each eigenfunction that is at most a polynomial of degree r in the spins has as the coefficient of the highest power of the spins an arbitrary homogeneous function of the momentum vectors. If this coefficient function satisfies $\lambda^q f(k_1,k_2,\ldots,k_r)=f(\lambda k_1,\lambda k_2,\ldots,\lambda k_r)$ then the eigenvalue of such an eigenfunction is $d+(2-d)\times(r/2-q)$.

Fully nonlinear problems must be considered individually. For use in Secs. II and III we consider a system of spin dimension $n+m$, which breaks into two internally isotropic blocks; that is, the Hamiltonian is a function of $x\equiv\sum_{i=1}^{n}s_i^2$ and $y\equiv\sum_{i=n+1}^{n+m}s_i^2$ alone. The WHARG equations are

$$\dot{H} = dH + (2-d)\left[x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y}\right] + \frac{d}{2}\left[n\ln\left(1+\frac{\partial H}{\partial x}\right) + m\ln\left(1+\frac{\partial H}{\partial y}\right) + \ln\left(1 + \frac{2x\,\partial^2 H/\partial x^2}{1+\partial H/\partial x} + \frac{2y\,\partial^2 H/\partial y^2}{1+\partial H/\partial y} + \frac{4xy[(\partial^2 H/\partial x^2)(\partial^2 H/\partial y^2) - (\partial^2 H/\partial x \partial y)^2]}{(1+\partial H/\partial x)(1+\partial H/\partial y)}\right)\right] . \tag{A9}$$

It is more compact to leave the nonlinear structure inside the logarithm in this case. Now we write H as

$$H = r_x x + r_y y + u_{xx}\tfrac{1}{2}x^2 + u_{yy}\tfrac{1}{2}y^2 + u_{xy}xy , \tag{A10}$$

and make the change of variables:

$$\begin{aligned} r_n &\equiv r_x/(1+r_x) , \\ r_m &\equiv r_y/(1+r_y) , \\ u_n &\equiv u_{xx}/(1+r_x)^2 , \\ u_m &\equiv u_{yy}/(1+r_y)^2 , \\ w &\equiv u_{xy}/(1+r_x)(1+r_y) . \end{aligned} \tag{A11a}$$

Then, the WHARG equations are

$$\begin{aligned} \dot{r}_n &= (1-r_n)\left(2r_n + \frac{d}{2}[(n+2)u_n + mw]\right) , \\ \dot{r}_m &= (1-r_m)\left(2r_m + \frac{d}{2}[(m+2)u_m + nw]\right) , \\ \dot{u}_n &= \epsilon u_n - \frac{d}{2}[(n+8)u_n^2 + mw^2] \\ &\quad - 4u_n\left(r_n + \frac{d}{4}[(n+2)u_n + mw]\right) , \\ \dot{u}_m &= \epsilon u_m - \frac{d}{2}[(m+8)u_m^2 + nw^2] \\ &\quad - 4u_m\left(r_m + \frac{d}{4}[(m+2)u_m + nw]\right) , \\ \dot{w} &= \epsilon w - \frac{d}{2}w[4w + (n+2)u_n + (n+2)u_m] \\ &\quad - 2w\left(r_n + r_m + \frac{d}{4}[(n+m)w + (n+2)u_n + (m+2)u_m]\right) . \end{aligned} \tag{A11b}$$

It is easy to check that coefficients of higher powers of the spin are $O(\epsilon^3)$. These equations are already diagonalized around the infinite Gaussian point, $r_n = r_m = 1$, $u_n = u_m = w = 0$. Diagonalizing around the finite Gaussian point $r_n = r_m = u_n = u_m = w = 0$, we make the further change of variables:

$$\begin{aligned} x_n &= r_n + \frac{d}{2(2-\epsilon)}[(n+2)u_n + mw] , \\ x_m &= r_m + \frac{d}{2(2-\epsilon)}[(m+2)u_m + nw] , \end{aligned} \tag{A12}$$

and write the WHARG equations as (neglecting terms like $\epsilon^2 u_n^2$ and $\epsilon^2 u_m^2$)

$$\begin{aligned} \dot{x}_n &= 2x_n[1 - x_n - (n+2)u_n] - 2mx_m w , \\ \dot{x}_m &= 2x_m[1 - x_m - (m+2)u_m] - 2nx_n w , \\ \dot{u}_n &= \epsilon u_n - \frac{d}{2}[(n+8)u_n^2 + mw^2] - 4u_n x_n , \\ \dot{u}_m &= \epsilon u_m - \frac{d}{2}[(m+8)u_m^2 + nw^2] - 4u_m x_m , \\ \dot{w} &= \epsilon w - \frac{d}{2}w[4w + (n+2)u_n + (m+2)u_m] - 2w(x_n + x_m) . \end{aligned} \tag{A13}$$

The equations given in (A13) do not lend themselves easily to global analysis. There are 32 fixed points of (A13), many of which are not particularly interesting. Points of particular interest in the three-dimensional subspace $x_n = x_m = 0$ are the Gaussian point at $u_n = u_m = w = 0$; the n-spin point at $u_m = w = 0$, $u_n = \epsilon/2(n+8)$; the m-spin point at $u_n = 0 = w$, $u_m = \epsilon/2(m+8)$; the coexisting but uncoupled (n, m) spin point at $w = 0$, $u_n = \epsilon/2(n+8)$, $u_m = \epsilon/2(m+8)$; and the $n+m$ spin point at $w = u_n = u_m = \epsilon/2(n+m+8)$. The usual n and m spin points are unstable with respect to the uncoupled (n, m) point which is unstable with respect to the $n+m$ spin point. Thus, if these fixed points were sufficient to describe the system, the $n+m$ spin point would be the most stable.

However, there are also fixed points for nonzero x_n and x_m. The most important are the "isolated" n-spin point at $x_n = 0$, $x_m = 1(!)$, $u_m = w = 0$, $u_n = \epsilon/2(m+8)$ and the "isolated" m-spin point at $x_m = 0$, $x_n = 1(!)$, $u_n = w = 0$, $u_m = \epsilon/2(m+8)$. Recalling that x_n (respectively, x_m) = 1 implies that $r_n = \infty$ (respectively, $r_m = \infty$), we see that these isolated points correspond to systems for which one subsystem is at its critical temperature while the other subsystem is effectively at "infinite" temperature. It is clear that no real Hamiltonian can be said to be "at" these fixed points. However, the $n+m$ isotropic-spin fixed point, which is the stablest of the $x_n = x_m = 0$ fixed points is unstable with respect to these isolated n-spin and m-spin points. These isolated points are unstable only with respect to the infinite Gaussian fixed point $x_n = x_m = 1$, $u_n = u_m = w = 0$. The full five-dimensional space is thus partitioned in such a way that one four-dimensional manifold is generated by the isolated m-spin fixed point; a second four-dimensional manifold is generated by the isolated n-spin point; these manifolds

intersect in the three-dimensional manifold $x_n = x_m = 0$, which is generated by the $n+m$ spin point (by generated we mean that the fixed point in question is a stable node with respect to all trajectories lying entirely within the manifold, that is, the fixed point is the stablest fixed point contained within the manifold.) The fieldlike variable carrying the system from one of the four-dimensional manifolds through the three-dimensional boundary to the other four-dimensional manifold is essentially the anisotropy field. Linearizing about the $n+m$ spin fixed point gives the usual determination of the crossover exponent,[13] $\varphi = 1 + \epsilon(n+m)/2(n+m+8)$.

The complicated geometry of the four-dimensional manifolds precludes any direct attack on the full five-dimensional problem. For this reason, Sec. III of the text considers the special case of $n = m$ and $x_n = x_m$, $u_n = u_m$. Specializing equations (A11)–(A13) to this case, one obtains (3.2).

APPENDIX B: CALCULATION OF Φ AND $\mathcal{Y}$

In this appendix we will discuss the calculation of the two-parameter separatrix and crossover functions Φ and $\mathcal{Y}$ which are used in both Secs. II and III. Consider the coupled first-order differential equations,

$$\dot{x} = 2x(1 - x - \tfrac{1}{2}\epsilon\Delta y)\,, \tag{B1}$$

$$\dot{y} = y[\epsilon(1-y) - 4x]\,. \tag{B2}$$

We are interested in the form of the separatrix $y = \Phi(x)$, joining the point $x = 0$, $y = 1$ to the point $x = 1$, $y = 0$ (cf. Fig. 1). We may form an equation for $\Phi(x)$ by noting that $\Phi'(x)\,\dot{x}(x,\Phi) = \dot{y}(x,\Phi)$ or

$$\Phi'(x)\,2x(1 - x - \epsilon\Delta\Phi/2) = \Phi[\epsilon(1-\Phi) - 4x]\,. \tag{B3}$$

Near $x = 1$ we expect $\Phi \ll 1 - x$. Examining (B3) in that limit, it reduces to the linear equation

$$2\Phi'(1-x) = -d\Phi\,, \tag{B4}$$

which has solutions proportional to $(1-x)^{d/2}$. If we set $\Phi = (1-x)^{d/2}\,e^{g}$ we obtain an equation for g. This equation is rather messy and it is better within the context of the WHARG equations to perform an ϵ expansion. A form for g which contains all $O(\epsilon)$ corrections exactly is

$$\Phi = (1-x)^{d/2}\exp[\tfrac{1}{2}\epsilon x(1-2\Delta)]\,. \tag{B5}$$

However, since this is arrived at in terms of an expansion in ϵ, it is not immediately clear that (B5) is the best or most suitable way of writing the solution of (B3) to $O(\epsilon)$. We have checked the suitability of such an expansion to show that no extraneous singularities would be introduced at that order.

We now turn our attention to $\mathcal{Y} \equiv \exp(-\int y\,dl)$. On the line $x = 0$, $\mathcal{Y} = 1 - y$ by explicit integration. We also recall that $\mathcal{Y}$ vanishes on the separatrix $y = \Phi(x)$. We therefore try a solution of the form $\mathcal{Y} = (1 - y/\Phi)\,e^{g\mathcal{Y}}$. To first order in ϵ, we find

$$\mathcal{Y} = (1 - y/\Phi)\exp(\epsilon\Delta xy/\Phi)\,. \tag{B6}$$

We have also checked that this form is suitable.

Thus we have shown that the forms given in (B5) and (B6) are suitable $O(\epsilon)$ approximate solutions. Since they are used in a variety of contexts in Secs. II and III we repeat their expressions here and display their dependence on Δ as an additional argument.

$$\Phi(x,\Delta) = (1-x)^{d/2}\exp[\tfrac{1}{2}\epsilon x(1-2\Delta)]\,, \tag{B7}$$

$$\mathcal{Y}(x,y,\Delta) = [1 - y/\Phi(x,\Delta)]\exp(\epsilon\Delta xy/\Phi)\,. \tag{B8}$$

It is important to stress that (B7) and (B8) give the solutions to (B1) and (B2) to $O(\epsilon)$. Thus, in Sec. II, the solution given in (2.9) is the proper $O(\epsilon)$ solution of (2.5). However, it is not a proper solution to (2.3), except formally. If the solution given in (2.9) is tested with the transformations given in (2.3) one of the $O(\epsilon^2)$ terms which is formally dropped is proportional to $\epsilon^2 y_n^2/x$. This is to be expected since (2.3) and (2.4) have different locations for their singularities. In passing to (2.4), the separatrix connecting the finite Gaussian and WF points was approximated by a straight line. It is easy to see that if terms up to y_n^m are kept in the definition of x ($x = 0$ defines the critical separatrix) the terms to be formally discarded include one proportional to $\epsilon^{m+1}y_n^{m+1}/x$. Thus to avoid this inconvenient singularity in the solution of (2.3) we would have to go to arbitrary order in ϵ. Although we will argue below that such an effort is bootless, we will sketch briefly the results of such a solution.

We make the exact change of variable $x = \gamma + \rho(\epsilon y_n)$ in (2.3) and demand that $x = 0$ represent the $T = T_c$ separatrix. The resulting equations are

$$\dot{x} = 2x\left[1 - x - \tfrac{1}{2}\epsilon y_n\Delta_n + 2\left(\rho - y_n\frac{d\rho}{dy_n}\right)\right], \tag{B9}$$

$$\dot{y}_n = y_n\{\epsilon[1 - y_n(1+2\Delta_n)] - 4x + 4\rho\}\,, \tag{B10}$$

where ρ satisfies

$$0 = 2(1+\rho)\,(\tfrac{1}{2}\epsilon y_n\Delta_n - \rho) + y_n\frac{d\rho}{dy_n}\{\epsilon[1 - y_n(1+2\Delta_n)] + 4\rho\}\,. \tag{B11}$$

Matching the solution to (2.3), (B9) and (B10) will involve R, X, Y_n, and two new functions Y_n' and Y_n'' defined by $\dot{Y}_n' = 4\rho\,Y_n'$ and $\dot{Y}_n'' = 4y_n(d\rho/dy_n)\,Y_n''$, respectively. Y_n' and Y_n'' are simple powers of Y_n if the linear approximation for ρ is made as in (2.4). An immediate consequence of (B9)–(B11) is that the singularity of Y_n at $x = 0$ is changed from $(1 - y_n)$ to $(1 - y_n)^{1+a}$, where $a = 2\Delta_n - 4(d\rho/dy_n) \approx -\epsilon\Delta_n$.

The effect of keeping the curvature of the sepa-

ratrix is to introduce at least a cubic term into the y_n equation (B11). However we have already discarded from the y_n equation any six-spin interaction term which "feeds back" from the higher-order equations. By examining the WHARG equations one finds that the six-spin coupling constant is $O(y_n^3)$. Thus, we cannot keep any curvature in the separatrix without including the six-spin terms. If we were to use the exact separatrix we would have to solve the infinite set of WHARG equations. Further more, the use of the WHARG equations could not be justified since the momentum dependence has been neglected.

We also note that logrithmic corrections of the sort described by Wegner[17] do not appear in an ϵ-expansion. Reference 17 gives a general procedure to extract nonlinear scaling fields and shows that the method may fail if the eigenvalues satisfy certain integral relations. In the examples in Secs. II or III, Wegner would predict logrithmic corrections if $2/\epsilon = N$, an integer. These corrections will never appear in any ϵ expansion. The source of these terms is the vanishing of the denominator of some coefficient in a tentative power-series expansion for the scaling fields of the form $1/(N\epsilon - 2)$. In a consistent ϵ expansion this denominator must be expanded at $-\frac{1}{2} - \frac{1}{4}N\epsilon\ldots$. Thus, to any fixed order in ϵ no difficulty is encountered.

This rather unhappy resolution is closely related to a similar situation in the field-theoretic approach to the ϵ expansion. The Feyman diagram illustrated in Fig. 7(a) diverges as P^{-6} for small P ($T = T_c$). This divergence is not troublesome in a few simple diagrams, but by concatenating N such simple loops the divergence is increased to P^{-N6} [Fig. 7(b)]. The multiple loop diagram can replace the single loop diagram in any Feyman diagram, and no further "renormalization" removes this divergence. To handle this, a consistent ϵ expansion is performed so that $P^{-N\epsilon} = 1 - N\epsilon \ln P + \ldots$. The divergence of each log term is sufficiently weak to be incorporated into the remaining diagramatic calculations and renormalization procedure. Thus, in both cases a real singularity is removed by the use of a self-consistent ϵ expansion.

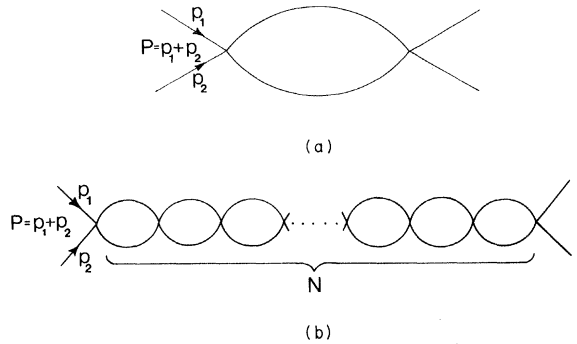

FIG. 7. Feynman diagrams for simple loop divergence. (a) A single loop with momenta $\vec{p}_1$ and $\vec{p}_2$ diverges like P^{-6} where $\vec{P} = \vec{p}_1 + \vec{p}_2$. (b) N such loops linked together give a diagram which diverges like P^{-N6}, which is of arbitrary order for sufficiently large N.

We also note that the expansion of the nonlinear scaling fields in power series[17] is limited in usefulness by the fact that the fields contain singularities. The zero of the function $\mathcal{Y}$, for example, is hard to locate in a power-series expansion. The delineation of the basic singularities of the nonlinear scaling fields enables the series expansions of Ref. 17 to be partially summed to give faithful representations of the scaling fields.

APPENDIX C: SPECIAL CASES $n = 0, \infty, 2, 4$

In a few special cases, the solution of the equations of Sec. III can be carried further.[46] For $n = 0$ and $n = \infty$ completely explicit solutions can be obtained within the context of an ϵ expansion. For $n = 2$ and $n = 4$, the renormalization trajectories in the $x = 0$ plane can be obtained exactly.

For $n = 0$ and $n = \infty$, we may take $z = y_n + y_{2n}$. The x equation and the z equation decouple from the remaining equation. We have

$$\dot{x} = 2x(1 - x - \tfrac{1}{8}\epsilon z/8), \tag{C1}$$

for $n = 0$, and

$$\dot{x} = 2x(1 - x - \tfrac{1}{2}\epsilon z/2), \tag{C2}$$

for $n = \infty$; while in both cases the z equation is

$$\dot{z} = z[\epsilon(1 - z) - 4x]\ . \tag{C3}$$

Combining the solution of the two-dimensional problem with the information given in (3.9) we have immediately for $n = 0$,

$$Y_{2n}^{1/2} = \frac{y_n + y_{2n} Y_0^{1/2}}{y_n + y_{2n}}\ , \tag{C4}$$

$$Y_n^{1/2} = \frac{Y_0^{1/2}(y_n + y_{2n})}{y_n + y_{2n} Y_0^{1/2}}\ , \tag{C5}$$

where $Y_0 = \mathcal{Y}(x, y_n + y_{2n}, \frac{1}{4})$. Similarly for $n = \infty$, we have

$$Y_n = \frac{y_n Y_\infty + y_{2n}}{y_n + y_{2n}}\ , \tag{C6}$$

$$Y_{2n} = \frac{Y_\infty(y_n + y_{2n})}{Y_\infty y_n + y_{2n}}\ , \tag{C7}$$

where $Y_\infty = \mathcal{Y}(x, y_n + y_{2n}, 1)$.

For the case $n = 2$ and $n = 4$ [where (3.9) fails to provide any information], some extra information is available in the form of the renormalization trajectories in the $x = 0$ plane. In principle, Y_n and

Y_{2n} can be obtained in the $x=0$ plane by one integration. The form of Y_n and Y_{2n} at $x=0$ is probably sufficient since we are interested in $x\to 0$ and we know how the boundary conditions for Y_n and Y_{2n} depend on x (see Sec. III). For $n=2$, the trajectories are given implicitly by

$$\left[\frac{y_{2n}^5}{y_n^4}-\left(\frac{y_{2n}}{y_n}\right)^4+\frac{4}{5}\left(\frac{y_{2n}}{y_n}\right)^3-\frac{12}{25}\left(\frac{y_{2n}}{y_n}\right)^2+\frac{24}{125}\frac{y_{2n}}{y_n}-\frac{24}{625}\right]e^{5y_{2n}/y_n}=k, \quad \text{(C8)}$$

while, for $n=4$, the corresponding equation is

$$\left[\frac{y_n^4}{y_{2n}^3}-\left(\frac{y_n^3}{y_{2n}}\right)+\frac{3}{4}\left(\frac{y_n}{y_{2n}}\right)^2-\frac{3}{8}\frac{y_n}{y_{2n}}+\frac{3}{32}\right]e^{4y_n/y_{2n}}=k' . \quad \text{(C9)}$$

APPENDIX D: ZERO-FIELD GIBBS POTENTIAL

As discussed in Sec. II, the zero-field Gibbs potential (for $T>T_c$) can be written as[11b,26,27]

$$G=\frac{dn}{2}\int_0^\infty e^{-dl}\ln[1+r(l)]\,dl \quad \text{(D1)}$$

In (D1) we have set $\eta=0$. Integrating this equation by parts we have that

$$G=\tfrac{1}{2}n\ln[1+r(0)]+\frac{n}{2}\int_0^\infty e^{-dl}\frac{\dot r(l)}{1+r(l)}\,dl . \quad \text{(D2)}$$

Examination of Eqs. (2.2) and (2.4) for the two-parameter problem or Eqs. (3.2) and (3.3) for the three-parameter problem shows that to $O(\epsilon)$ we may replace $\dot r/(1+r)$ by $2x$ and $(1+r)$ by X^{-1}. Equation (D2) now reads

$$G=-\tfrac{1}{2}n\ln X(0)+n\int_0^\infty e^{-dl}x(l)\,dl . \quad \text{(D3)}$$

The Gibbs potential can now in principle be evaluated by expression x in terms of the nonlinear scaling fields given in (2.10) and (3.14) for the two-parameter and three-parameter problems, respectively, and performing the integration in (D3).

For example, in the two-parameter case we may write the integral in (D3)

$$\int_0^\infty e^{-dl}x(l)\,dl=\tfrac{1}{2}S_n^{2-\alpha_n}(x,y_n)\times\int_x^1\frac{S_n^{\alpha_n-2}(x',y_n(x'))\,dx'}{1-x'-(\epsilon\Delta_n/2)y_n(x')} . \quad \text{(D4)}$$

In (D4), $y_n(x')$ denotes the value of y_n at $x=x'$ on the renormalization trajectory passing through (x,y_n). We may consider two limiting cases of (D4).

First we restrict (x,y_n) to lie on the separatrix $y_n=\varphi_n(x)$. In this case, using (2.9b) we have

$$\int_0^\infty e^{-dl}x(l)\,dl=\frac{1}{2}\frac{x^{2-\alpha_n}}{(1-x)^{d/2}}\times\int_x^1\frac{(1-x')^{d/2-1}}{(x')^{2-\alpha_n}}\frac{dx'}{1-\epsilon\Delta_n(1-x)/2} . \quad \text{(D5)}$$

As a second example we consider (x,y_n) such that $y_n\ll(1-x)^2$ all along the renormalization trajectory. In this case we may neglect Y_n and write

$$\int_0^\infty e^{-dl}x(l)\,dl=\frac{1}{2}\frac{x^{d/2}}{(1-x)^{d/2}}\int_x^1\frac{(1-x')^{d/2-1}}{x'^{d/2}}\,dx' . \quad \text{(D6)}$$

The condition $y_n\ll(1-x)^2$ cannot be satisfied as $x\to 0$ unless y_n is identically zero; in all other cases, the trajectories sweep toward the separatrix so that the free energy more closely approximates the solution given in (D5). We note that the prefactor of the integral in (D6) is simply $S_G^{2-\alpha_G}$ with Y_n set = 1.

For the general case, the trajectories $y_n(x')$ are given implicitly in the renormalization invariant (2.13). For further discussions and model calculations, see Refs. 11b, 26, and 27.

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