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The pinning paths of an elastic interface

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Abstract. – We introduce a Markovian model describing the paths that pin an elastic interface moving in a two-dimensional disordered medium. The scaling properties of these "elastic pinning paths" (EPP) are those of a pinned interface belonging to the universality class of the Edwards-Wilkinson equation with quenched disorder. We find that the EPP are different from paths embedded on a directed percolation cluster, which are known to pin the interface of the "directed percolation depinning" class of surface growth models. The EPP are characterized by a roughness exponent $\alpha = 1.25$, intermediate between that of the free inertial process ($\alpha = 3/2$) and the diode-resistor problem on a Cayley tree ($\alpha = 1$). We also calculate numerically the mean cluster size and the cluster size distribution for the EPP.

The problem of interface roughening in the presence of quenched disorder is a topic of recent interest, due to its importance as a paradigm in condensed-matter physics and due to the broad range of applications [1]. In a typical case, the interface moves in a $(d+1)$-dimensional disordered medium driven by a homogeneous force F. At small forces, the interface is pinned by the impurities of the medium, while the interface undergoes a depinning transition at a critical force F_c, and for $F > F_c$ the interface moves with a nonzero velocity. The spatial fluctuations of the interface are characterized by the scaling of the saturated interface width $W_{\mathrm{sat}}(L)$ defined as the rms fluctuations of the interface height for a system size L,

$$W_{\mathrm{sat}}(L) \sim L^{\alpha}, \tag{1}$$

where α is the roughness exponent.

It has been proposed [2], [3] that the depinning transition can be described by the following equation of motion for the interface height $y(\mathbf{x}, t)$:

$$\frac{\partial}{\partial t} y(\mathbf{x}, t) = \nabla^2 y + \eta(\mathbf{x}, y) + F, \tag{2}$$

where $\mathbf{x}$ is the d-dimensional coordinate parallel to the interface. The first term on the right-hand side of (2) represents the surface tension favoring a smooth interface, and we say that the interface is elastic. The second term is a random field that mimics the quenched disorder of the medium and is assumed to have zero mean and short-range correlations.

The universality class corresponding to eq. (2) is called quenched Edwards-Wilkinson (QEW), because (2) is similar to the Edwards-Wilkinson equation [4]. The difference, which changes the behavior of (2) drastically, is the presence of spatially dependent quenched disorder $\eta(\mathbf{x}, y)$ instead of time-dependent shot noise $\eta(\mathbf{x}, t)$. Numerical studies [5], [6] of the depinning transition yield a roughness exponent $\alpha \simeq 1.25$ in $d = 1$ ($(1 + 1)$ dimensions) and $\alpha \simeq 0.75$ for $d = 2$ ($(2 + 1)$ dimensions). These values are lower than the results of perturbation theory [2], which yields $\alpha = 3/2$ in $d = 1$ and $\alpha = 1$ for $d = 2$. On the other hand, the numerical values are significantly higher than the prediction of a functional renormalization group treatment which gives $\alpha \simeq 1$ and $\alpha \simeq 2/3$ for $d = 1$ and $d = 2$, respectively [7].

The relevance of directed percolation to interface depinning has been established for a different class of models, called *directed percolation depinning* models [8], which are in the same universality class as eq. (2) when a Kardar-Parisi-Zhang (KPZ) [9] term $\lambda(\nabla y)^2$ is included. In these models, the interface is pinned by paths on a directed percolation cluster of pinning sites [8]. Thus, the scaling properties of the interface at the depinning transition in $(1 + 1)$ dimensions can be obtained by a mapping onto directed percolation (DP) [10], from which a roughness exponent $\alpha \simeq 0.63$ is obtained.

In this paper, we investigate the scaling properties of the paths which pin the interface for the QEW universality class for the case $d = 1$. We term these paths elastic pinning paths (EPP). Our numerical results provide an independent check for the anomalous roughness exponent $\alpha \simeq 1.25$ obtained with the models of the QEW universality class. We also consider two known random walk models, which yield $\alpha = 3/2$ and $\alpha = 1$, respectively. By comparing the EPP to these random walks we obtain some insight into why the roughness exponent of eq. (2) lies in the interval $1 < \alpha < 3/2$.

To motivate the definition of the EPP, we first consider a discrete solid-on-solid model corresponding to eq. (2) [5]. The interface position h_i is defined on a square lattice of lateral size L. We assign to each site on the lattice a random number $\eta_{i,h}$ which can have two values, $\eta_{i,h} = 1$ (unblocked cell) with probability p, and $\eta_{i,h} = -1$ (blocked cell) with probability $1 - p$. A local force is defined by

$$f_i \equiv h_{i+1} + h_{i-1} - 2h_i + \eta_{i,h}. \tag{3}$$

At time $t = 0$ the interface is flat, and at a given time the height of the i-th column is increased by one if the local force f_i is positive.

At a critical value of the probability $p = p_\mathrm{c}$, the interface is pinned by one of the pinning paths. According to the dynamical rules, a pinned interface satisfies $f_i \leq 0$ for all i. We define the increments of a given path as $\Delta_i \equiv h_i - h_{i-1}$, so according to eq. (3) a spanning path stopping the QEW interface satisfies $\Delta_{i+1} \leq \Delta_i - \eta_{i,h}$. By induction, one can show that a lower bound for $\Delta_{i+1} - \Delta_i$ holds at every time step of the interface evolution $-2 \leq \Delta_{i+1} - \Delta_i$. Therefore, the possible pinning paths are defined by

$$-2 \leq \Delta_{i+1} - \Delta_i < -\eta_{i,h}. \tag{4}$$

Note that paths that reach the same cell (i, h_i) can have different increments Δ_i. We start at $i = 1$ with $h_1 = 0$ and initial increment $\Delta_1 = 0$. The paths in column $i + 1$ are updated according to the following three rules:

Rule i): If the cell (i, h_i) is blocked ($\eta_{i,h} = -1$) the path is splitted into two new paths according to (4) [11], where the positions at $i + 1$ are $h_{i+1} = h_i + \Delta_i \pm 1$.

Rule ii): If the cell (i, h_i) is unblocked ($\eta_{i,h} = 1$) the new position of the path is $h_{i+1} = h_i + \Delta_i - 1$ [11].

Rule iii): The path stops when $h_i \leq 0$.

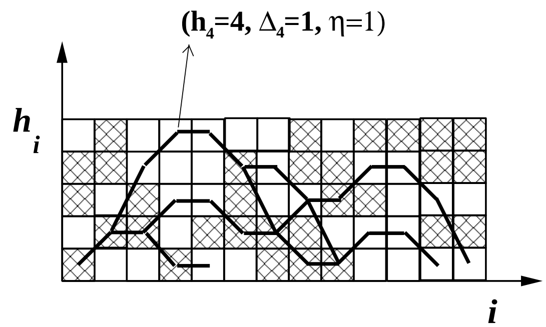

Fig. 1. – Example of one cluster of pinning paths generated by the proposed rules. The sites in a square lattice are unblocked ($\eta = 1$, shown in white) with probability p and blocked ($\eta = -1$, shown as cross-hatched) with probability $1 - p$. The path starts at $x_1 = 0$ with $h_1 = 0$ and $\Delta_1 = 0$ and we apply the rules *i*), *ii*), and *iii*).

After moving to the new cell the increment Δ_{i+1} is updated and the rules are applied again. Rules *i*) and *ii*) are the implemetation of eq. (4). Rule *iii*) is motivated by the fact that if the path deviates too much in the downward direction, it would not have a chance to block the growth since, in a system with periodic boundary conditions, the path should return to the same point where it starts. In fig. 1 we show a typical set ("cluster") of directed paths. The paths are characterized by large local slopes which is the main feature of the QEW interface at the depinning transition.

The scaling properties of the directed paths require two characteristic lengths, $\xi_\parallel$ and $\xi_\perp$, the correlation length parallel and perpendicular to the preferred direction of the paths [10]. The correlation lengths diverge at the critical concentration of pinning centers p_c as $\xi_\parallel(p) \sim |p - p_c|^{-\nu_\parallel}$, and $\xi_\perp(p) \sim |p - p_c|^{-\nu_\perp}$, where $\nu_\parallel$ and $\nu_\perp$ are two different universal exponents.

A cluster consisting of pinning paths is defined by all paths generated by the rules *i*)-*iii*) for one realization of the disorder. Let us denote by s the number of sites in a given cluster. The cluster size distribution $n_s(p)$, defined as the average number of clusters of s sites per lattice site, shows a power law behavior at p_c. For $p < p_c$ only finite clusters are present, so there exists an effective cut-off for the cluster size, $s_0 \sim |p - p_c|^{-1/\sigma}$. The cluster size distribution

Table I. – *Critical exponents for the* EPP *presented in this paper along with the results for the elastic interface in the* QEW *universality class, and the* DP *universality class. We find* $p_c = 0.282 \pm 0.001$ *for the* EPP.

	EPP	QEW (interface)	DP
α	1.26 ± 0.03	1.25 ± 0.01 [5]	0.63 ± 0.04 [8]
$\nu_\parallel$	1.33 ± 0.04	1.35 ± 0.04 [6]	1.73 ± 0.01 [17]
$\nu_\perp$	1.67 ± 0.04	1.68 ± 0.04	1.09 ± 0.01 [17]
τ	1.43 ± 0.02	—	1.28 ± 0.02 [17]
γ	2.43 ± 0.05	—	2.28 ± 0.01 [17]

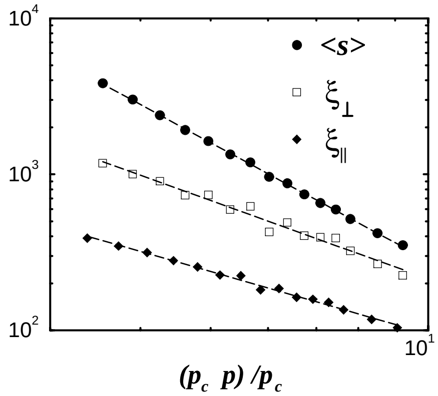

Fig. 2

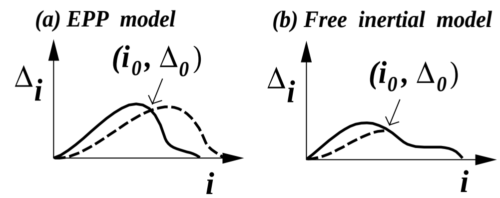

Fig. 3

Fig. 2. – Log-log plots of the mean cluster size $\langle s \rangle$ (shifted for clarity), the correlation lengths $\xi_\perp$ and $\xi_\parallel$ in the perpendicular and parallel directions as a function of the reduced probability $(p_c - p)/p_c$. Simulations are averaged over 10^7 clusters.

Fig. 3. – Schematic illustration of the topmost trajectory in the (i, Δ_i)-plane for (a) the EPP, and (b) the modification of the EPP where the noise is chosen according to the position of the path in the (i, Δ_i)-plane (free inertial process). In both figures we plot two paths (solid and dashed lines) which intersect at the point (i_0, Δ_0). In the EPP case the paths can cross each other, because the noise is determined by the position in real space (i, h_i). Thus, also paths in real space can cross each other. In the free inertial process, however, if two paths intersect at (i_0, Δ_0) then they continue together. Therefore, paths in real space cannot cross each other.

has the scaling form $n_s(p) \sim s^{-\tau}\ g(s/s_0)$, where $g(x)$ is a scaling function that decreases faster than a power law for $x \gg 1$. The mean cluster size $\langle s \rangle$ also diverges at p_c as a power law $\langle s \rangle \sim |p - p_c|^{-\gamma}$, with $\gamma = (2 - \tau)/\sigma$ [10].

The scaling relations presented so far are valid for infinite lattices. Finite-size scaling considerations allow us to write [8] $W(L) \sim \xi_\perp \sim \xi_\parallel^{\nu_\perp/\nu_\parallel} \sim L^{\nu_\perp/\nu_\parallel}$. Hence from (1)

$$\alpha = \nu_\perp/\nu_\parallel. \tag{5}$$

In our simulations we compute all the exponents characterizing the scaling behavior of the EPP. Results are shown in fig. 2. We calculate the correlation length exponents and find $\nu_\parallel \simeq 1.33$ and $\nu_\perp \simeq 1.67$ (see table I). Using (5) we find $\alpha \simeq 1.25$ in agreement with the exponent found for the QEW interface [5], [6]. The study of the mean cluster size yields an exponent $\gamma \simeq 2.43$, while the exponent of the cluster size distribution is $\tau \simeq 1.43$.

Next we calculate the scaling of the mean square fluctuations or "width" of the paths defined as $\langle h_i^2 \rangle^{1/2}$ as a function of the parallel coordinate i calculated at p_c. We find $\langle h_i^2 \rangle^{1/2} \sim i^\alpha$ with $\alpha \simeq 1.27$, in agreement with the exponent $\alpha = \nu_\perp/\nu_\parallel = 1.25$ that we find using (5) and our numerical results for $\nu_\perp$ and $\nu_\parallel$ [12].

Next, we discuss the relation of the EPP to two known universality classes —which can be considered as random walk models— and to which the EPP can be modified by changing the interaction with the disorder. The first random walk model is related to the free inertial process studied in [13]. Here, we assume that the noise is determined by the position (i, Δ_i) in the increment space (η_{i,Δ_i}), instead of being determined by the position (i, h_i) in the real space as for the EPP model. Since the average span of $\Delta_i \sim i^{\alpha'}$ is much smaller than the average span of $h_i \sim i^{\alpha}$ (*i.e.* $\alpha' = \alpha - 1 < \alpha$), many of the EPP that were treated differently (since they had different h_i coordinate, and therefore different interaction with the noise) now become indistinguishable since many of them can have the same value of Δ_i. Therefore the number of different paths decreases significantly compared to the EPP case.

The span of the cluster is determined by the topmost trajectory in real space which is in turn represented by the topmost trajectory in the increment space (i, Δ_i). Since the noise is chosen according to the position in the (i, Δ_i) space, two paths which arrive to the same point (i_0, Δ_0) remain together because also the noise is the same. Thus, two given paths cannot cross each other in the plane (i, Δ_i) (see fig. 3) and the topmost trajectory in the increment space is composed by a unique and well-defined path. This path is a simple random walk in the (i, Δ_i) plane, which has a certain probability of going up and down depending on the noise. Therefore the average span scales as $\Delta_i \sim i^{\alpha'}$ with $\alpha' = 1/2$. With $\alpha = \alpha' + 1$ we get $\alpha = 3/2 > \alpha_{\mathrm{EPP}} = 1.25$. This is in agreement with the fact that the standard deviation of the free inertial process [13] increases as $i^{3/2}$.

The critical probability, p_{c}, now corresponds to the unbiased random walk, and can be readily computed analytically for the rules *i*) and *ii*). The critical probability is larger than the corresponding value in the EPP case, being $p_{\mathrm{c}} = 1/2$. The exponent τ now is related to the probability of the first return to the origin after some number of steps i. For the free inertial process this probability decays as $i^{-5/4}$ [14], which differs from the random walk result $i^{-3/2}$. This indicates that the topmost trajectory in the (i, Δ_i)-plane determines only the scaling of the span of the clusters, but other properties, such as the probability of first return to the origin, are not determined by the topmost trajectory.

For the EPP, where the noise is chosen from the position in real space (i, h_i), the topmost trajectory in the (i, Δ_i)-plane still determines the scaling properties of the span on the clusters. However, in this case, the topmost trajectory in the increment space can be composed by parts of several different paths (fig. 3). This is so because two paths can cross each other in the (i, Δ_i)-plane because the noise is determined by the position in the (i, h_i)-plane which can be different for the two paths at the crossing point (i_0, Δ_0) (fig. 3). This is the main difference between the EPP and the free inertial process.

A second universality class obtained from the EPP model by changing the properties of the interaction with the disorder is obtained by defining the value of the noise for each path independently. Even for two paths that meet at a point (i, h_i) with the *same* Δ_i, the values of the noise are chosen as independent of each other. Since now all paths are independent and there are no loops, this model can be exactly mapped —with respect to Δ_i and i— to the diode resistor Cayley tree problem solved in [15]. The i coordinate is identified with the time coordinate of the Cayley tree cluster which, in turn, corresponds to the chemical or minimum path in the longitudinal hyperplane of the Cayley tree cluster (see [16]), and has $\nu_{\parallel} = 1/2$ [15]. The coordinate Δ_i is the remaining transversal coordinate of the Cayley tree cluster and has $\nu_{\perp} = 0$ [15]. Then, we find $\alpha' = 0$, and $\alpha = \alpha' + 1 = 1 < \alpha_{\mathrm{EPP}} = 1.25$. We also find numerically $\alpha' = 0$. The critical probability p_{c} in this case can be computed analytically, giving $p_{\mathrm{c}} = 1/4$, smaller than the EPP case.

What can we learn from the comparison of the EPP to the two discussed random walk models? Since the EPP can cross each other in the (i, Δ_i)-plane, there is no unique topmost path but a topmost trajectory consists of several paths. By being at some point not the topmost path, a given path can "optimize" its way through the randomness in the sense that it visits more blocked cells. Thus, it can stay alive longer ($h_i > 0$) compared to the free inertial process where the unique topmost path determines the scaling properties. This means that the EPP has a smaller roughness exponent than the free inertial value of 3/2.

The other random walk model, where the noise of different paths is always independent, has a roughness exponent $\alpha = 1$ which is even lower than that of the EPP. In order to understand the reason, we note that the paths have many more possibilities than the EPP at each point. Thus, it is not surprising that there are always paths which stay alive even for large i. It seems reasonable that the scaling properties are dominated by these paths, causing a smaller roughness exponent.

To summarize, we introduce the EPP model to describe the paths that pin an elastic interface belonging to the QEW universality class of interface growth. The critical paths that are characterized by an exponent $\alpha_{\mathrm{EPP}} = 1.25$ describe the scaling properties of an elastic interface pinned by the quenched disorder of the medium. The roughness exponent of the EPP lies between that of the free inertial process with $\alpha = 1.5$ [13], and that of the diode resistor Cayley tree problem with $\alpha = 1$ [15]. The comparison of the EPP with these two random walk models sheds some light on the value of $\alpha = 1.25$.

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REFERENCES

[1] Family F. and Vicsek T. (Editors), *Dynamics of Fractal Surfaces* (World Scientific, Singapore) 1991; Barabási A.-L. and Stanley H. E., *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge) 1995.

[2] Feigel'man M. V., *Sov. Phys. JETP*, **58** (1983) 1076.

[3] Bruinsma R. and Aeppli G., *Phys. Rev. Lett.*, **52** (1984) 1547; Koplik J. and Levine H., *Phys. Rev. B*, **32** (1985) 280.

[4] Edwards S. F. and Wilkinson D. R., *Proc. R. Soc. London, Ser. A*, **381** (1982) 17.

[5] Leschhorn H., *Physica A*, **195** (1993) 324.

[6] Makse H. A. and Amaral L. A. N., *Europhys. Lett.*, **31** (1995) 379.

[7] Nattermann T. *et al.*, *J. Phys. II*, **2** (1992) 1483; Narayan O. and Fisher D. S., *Phys. Rev. B*, **48** (1993) 7030.

[8] Tang L.-H. and Leschhorn H., *Phys. Rev. A*, **45** (1992) R8309; Buldyrev S. V. *et al.*, *Phys. Rev. A*, **45** (1992) R8313; Tang L.-H. and Leschhorn H., *Phys. Rev. Lett.*, **70** (1993) 3832; Amaral L. A. N., Barabási A.-L. and Stanley H. E., *Phys. Rev. Lett.*, **73** (1994) 62; Tang L.-H., Kardar M. and Dhar D., *Phys. Rev. Lett.*, **74** (1995) 920.

[9] Kardar M., Parisi G. and Zhang Y.-C., *Phys. Rev. Lett.*, **56** (1986) 889.

[10] Kinzel W., in *Percolation Structures and Processes*, edited by G. Deutscher, R. Zallen and J. Adler (A. Hilger, Bristol) 1983.

[11] According to (4), the implementation of rule i) would give four choices for the next position of the path $\Delta_{i+1} = \Delta_i + k$, $k = 1, 0, -1, -2$, and two choices for rule ii): $\Delta_{i+1} = \Delta_i + k$, $k = -1, -2$. However, we have checked that the rules can be simplified to the rules presented in the text without changing the universality class. These changes are analogous to the change of coordination numbers in directed percolation which change the value of the critical threshold but do not change the universality class [10].

[12] We also compute the mean square fluctuations of the increments Δ_i. We find $\langle \Delta_i^2 \rangle^{1/2} \sim i^{\alpha'}$ where $\alpha' = \alpha - 1$ because h_i is the integral of Δ_i along the path, so by integration the exponent α is $1 + \alpha'$.

[13] Masoliver J., *Phys. Rev. A*, **45** (1992) 706; Masoliver J. and Porra J. M., *Phys. Rev. Lett.*, **75** (1995) 189.

[14] Goldman M., *Ann. Math. Stat.*, **42** (1971) 2150.

[15] Buldyrev S. *et al.*, *Phys. Rev. E*, **52** (1995) 373.

[16] Havlin S. *et al.*, *Phys. Rev. Lett.*, **74** (1995) 4205.

[17] Essam J. W., Guttmann A. J. and De'Bell K., *J. Phys. A*, **21** (1988) 3815; Hede B., Kertész J. and Vicsek T., *J. Stat. Phys.*, **64** (1991) 829.