Breakdown of interdependent directed networks

Xueming Liu[a,b], H. Eugene Stanley[b,1], and Jianxi Gao[c,1]

[a]Key Laboratory of Image Information Processing and Intelligent Control, School of Automation, Huazhong University of Science and Technology, Wuhan 430074, Hubei, China; [b]Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215; and [c]Center for Complex Network Research and Department of Physics, Northeastern University, Boston, MA 02115

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Increasing evidence shows that real-world systems interact with one another via dependency connectivities. Failing connectivities are the mechanism behind the breakdown of interacting complex systems, e.g., blackouts caused by the interdependence of power grids and communication networks. Previous research analyzing the robustness of interdependent networks has been limited to undirected networks. However, most real-world networks are directed, their in-degrees and out-degrees may be correlated, and they are often coupled to one another as interdependent directed networks. To understand the breakdown and robustness of interdependent directed networks, we develop a theoretical framework based on generating functions and percolation theory. We find that for interdependent Erdős–Rényi networks the directionality within each network increases their vulnerability and exhibits hybrid phase transitions. We also find that the percolation behavior of interdependent directed scale-free networks with and without degree correlations is so complex that two criteria are needed to quantify and compare their robustness: the percolation threshold and the integrated size of the giant component during an entire attack process. Interestingly, we find that the in-degree and out-degree correlations in each network layer increase the robustness of interdependent degree heterogeneous networks that most real networks are, but decrease the robustness of interdependent networks with homogeneous degree distribution and with strong coupling strengths. Moreover, by applying our theoretical analysis to real interdependent international trade networks, we find that the robustness of these real-world systems increases with the in-degree and out-degree correlations, confirming our theoretical analysis.

interdependent networks | directed networks | degree correlations | percolation theory

The interdisciplinary field of networks has attracted the attention of scientists and engineers studying such wide-ranging topics as power systems (1), computer science (2, 3), biology, and social science (4, 5). Real-world network systems exhibit a high degree of heterogeneity, a fat-tailed degree distribution, and are scale-free (SF) (1). One important property of SF networks is that they are significantly more robust against random failure than classic Erdős-Rényi (ER) networks (6). The robustness of a network (7, 8) is its ability to continue functioning after experiencing targeted attacks or random failures. These can be characterized by using percolation theory to analyze the critical thresholds (8, 9) or be defined using the integrated size of the largest connected component during the attack period (7).

It is increasingly clear that almost all real-world critical infrastructures interact with one another. This has led to an emerging new field in network science that focuses on what are variously called interdependent networks, interconnected networks, a network of networks, multilayer networks, and multiplex networks (10–14) (*SI Appendix*, section I and *SI Appendix*, Fig. S1 compare these related designations). In these systems, networks interact with one another and exhibit structural and dynamical features that differ from those observed in isolated networks. For example, they may exhibit first-order phase transitions not only in their percolation (15) but also in their synchronization (16), and they may also have phase transitions that begin as first order, become hybrid, and then become second order (17). In interdependent networks the failure of a node in one network leads to the failure of the dependent nodes in other networks, which in turn may cause further damage to the first network, leading to cascading failures and possible catastrophic consequences. One example of cascading failure is the electrical blackout that affected much of Italy in 2003 (18) caused by a breakdown in two interdependent systems: the communication network and the power grid.

Mathematical frameworks have been proposed for analyzing the cascading failures in a pair of fully interdependent networks (15), a pair of partially interdependent networks (19), and a network of interdependent networks (20–22). These previous studies have been focused on networks that are undirected. Real-world networks, on the other hand, have directionality. Examples include metabolic networks and gene regulatory networks in biological systems (23), transportation networks and power grids in infrastructure systems (24, 25), and citation networks and trust networks in social systems (26). Directed multiplex networks with a giant strongly connected component (GSCC) are significantly more vulnerable than those with a giant weakly connected component (GWCC) (27).

The correlation between in-degree and out-degree is an important characteristic of directed network structure. In complex networks various aspects of this have been the focus of much study including robustness (28, 29), controllability (30), and synchronization (31). Much current research focuses on the in-degree and out-degree correlations of the dependency links between networks. Ref. 32 shows that the stability of a system relies on the in-degree and out-degree correlations between the coupled nodes in interdependent networks because these correlations curtail the tendency to catastrophic failure caused by the

Significance

Real-world complex systems interact with one another, and these interactions increase the probability of catastrophic failure. Using interdependent networks to model these phenomena helps understand a system's robustness and enables design of more robust infrastructures. Previous research has been limited to an idealized case where each layer is undirected, but almost all real-world networks are directed and exhibit in-degree and out-degree correlations. Therefore, we develop a general theoretical framework for analyzing the breakdown of interdependent directed networks with, or without, in-degree and out-degree correlations, and apply it to real-world international trade networks. Surprisingly, we find that the robustness of interdependent heterogeneous networks increases, whereas that of interdependent homogeneous networks with strong coupling strengths decreases with in-degree and out-degree correlations.

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[1]To whom correspondence may be addressed. Email: hes@bu.edu or jianxi.gao@gmail.com.

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interdependence. Still needed is an adequate theoretical analysis of cascading failures and robustness in interdependent networks where each network itself is a directed network with in-degree and out-degree correlations. Understanding how these two fundamental properties (the directionality and the in-degree and out-degree correlation) affect system robustness enables design of robust interdependent systems.

We develop here a theoretical framework for analyzing the breakdown of interdependent directed networks with and without in-degree k_{in} and out-degree k_{out} correlations, measured by a coefficient α with the form $k_{\text{out}} \propto k_{\text{in}}^{\alpha}$ (32, 33), and we apply it to real interdependent international trade networks. Our analysis leads to several unexpected results:

i) An interdependent system of directed networks is more vulnerable than coupled undirected networks, and an interdependent system of directed ER networks exhibits a hybrid phase transition that is not observed in coupled undirected ER networks (19).

ii) When comparing the robustness of interdependent SF networks with and without in-degree and out-degree correlations, we find that two criteria used to quantify the robustness—the critical threshold and the integral of giant component sizes—produce opposite results for some parameters.

iii) We find that in-degree and out-degree correlations in each layer could increase the robustness of interdependent networks with heterogeneous degree distributions, but decrease the robustness of interdependent networks with homogeneous degree distributions and with strong coupling strengths.

iv) Our theoretical analysis provides a framework for understanding the robustness of international trade networks. We find that an interdependent international trade network system is less robust than a system of randomized ER networks, produced by keeping the number of nodes and the number of links unchanged, and turning the real networks into directed ER networks. Furthermore, the in-degree and out-degree correlations in real networks increase the system's robustness.

Model

Fig. 1 shows a system of two interdependent directed networks, network A and network B, respectively, consisting of N_A and N_B nodes and following joint degree distributions $P_A(k_{\text{in}}, k_{\text{out}})$ and $P_B(k_{\text{in}}, k_{\text{out}})$, where k_{in} and k_{out} are the in-degree and out-degree of a given node, respectively. There are q_A fraction of nodes in network A (A nodes) that depend on the nodes in network B (B nodes), and q_B fraction of B nodes that depend on the A nodes. In addition, the nodes from the two networks are coupled following the no-feedback condition (20) that a node from one network can depend on no more than one node from the other network, and if node A_i depends on node B_j, node B_j depends on node A_k, then $i = k$. A node from one network stops functioning when the node from the other network on which it depends fails.

Cascading Failures and Percolation Process

We begin by randomly removing a fraction $1 - p_1$ of A nodes and a fraction $1 - p_2$ of B nodes. Each network fragments into strongly connected components, within which each pair of nodes can reach each other by a directed path. Only the GSCC (28) are assumed to be potentially functional. Thus, after the initial node removal, denoted by step $t = 1$, the fraction of functional A nodes and functional B nodes (in the GSCC) is, respectively, $\psi_1^{(s)} = p_1 p_A(p_1)$ and $\phi_1^{(s)} = p_2 p_B(p_2)$, where $p_A(x)$ and $p_B(x)$ are the formulas for calculating the sizes of GSCC in isolated network A and isolated network B, respectively, with x being an arbitrary complex variable (*SI Appendix*, section III.A). Because of the dependence between networks, the A nodes that depend on the failed B nodes fail, so at step $t = 2$, the fraction of functional

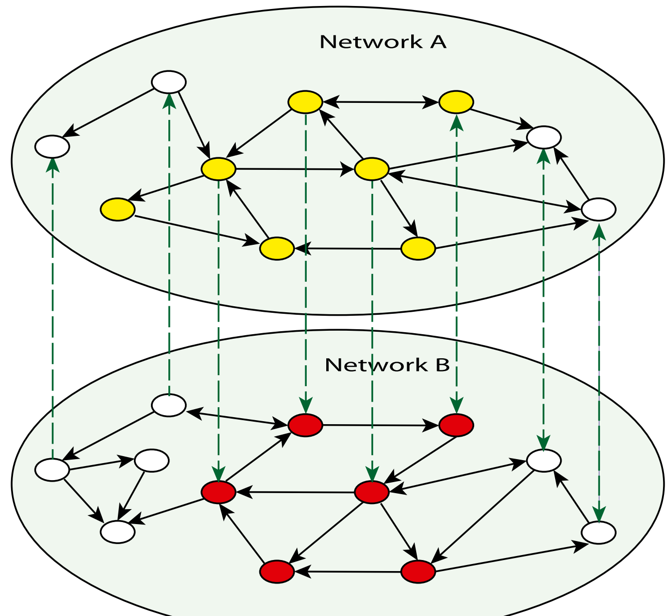

Fig. 1. Schematic demonstration of interdependent directed networks and the final GSCC. Directed networks A and B are coupled by directed dependency links (dotted lines) with a no-feedback condition (20). A dotted directed line from node *i* in one network to node *j* in the other network indicates that a failure of node *i* will cause node *j* to fail. The yellow nodes (network A) and red nodes (network B) indicate the final GSCC at the completion of cascading failure.

A nodes is $\psi_2^{(s)} = \psi_2' p_A(\psi_2')$, where $\psi_2' = p_1(1 - q_A(1 - p_B(p_1)p_2))$. Accordingly, any B nodes that depend on the failed A nodes also fail, and the fraction of functional B nodes is $\phi_2^{(s)} = \phi_2' p_B(\phi_2')$, where $\phi_2' = p_2(1 - q_B(1 - p_A(\psi_2')p_1))$. This process will iterate back and forth (will be a cascading failure) until there is no further damage from one network on the other, and the system goes into its steady state $\psi_\infty^{(s)}$ and $\phi_\infty^{(s)}$ (*SI Appendix*, section III.B). Fig. 1 shows the final GSCC of the two interdependent networks to be $\psi_\infty^{(s)} = 7/11$ (yellow) and $\phi_\infty^{(s)} = 6/12$ (red).

For the symmetric case with coupling strengths $q_A = q_B = q$, nonremoved nodes after initial failure $p_1 = p_2 = p$, and degree distributions $P_A(k_{\text{in}}, k_{\text{out}}) = P_B(k_{\text{in}}, k_{\text{out}}) = P(k_{\text{in}}, k_{\text{out}})$, whose generating function is $\Phi(x,y) = \sum_{k_{\text{in}}, k_{\text{out}}}^{\infty} P(k_{\text{in}}, k_{\text{out}}) x^{k_{\text{in}}} y^{k_{\text{out}}}$, the final GSCC size in both networks are the same $\psi_\infty^{(s)} = \phi_\infty^{(s)} \equiv p_\infty^{(s)}$, and follow

$$\frac{(1 - \Phi_1(z,1))\left(1 - q + \sqrt{(1-q)^2 + 4qp_\infty^{(s)}}\right)}{2(1-z)} = \frac{1}{p} \equiv R^{(s)}(z,q), \qquad [1]$$

where $\Phi_1(x,1) = \partial_y \Phi(x,y)\big|_{y=1} / \partial_y \Phi(1,1)$ is the generating function of a branching process (28).

For two interdependent directed ER networks with the same average degree $\langle k \rangle$, the final GSCC size is $p_\infty^{(s)} = (1-z)(1 - e^{\langle k \rangle / 2(z-1)})$. Fig. 2*A* shows their percolation behaviors with the fraction of remaining nodes p varying from 1 to 0 under three different coupling strengths. For an SF network, its in- and out-degree distributions follow $P(k_{\text{in}}) \propto k_{\text{in}}^{-\lambda_{\text{in}}}$ and $P(k_{\text{out}}) \propto k_{\text{out}}^{-\lambda_{\text{out}}}$, respectively. Here we set the parameter $\lambda_{\text{in}} = \lambda_{\text{out}} = \lambda$. The systems of interdependent SF networks with in-degree and out-degree correlations (Fig. 2 *B* and *C*) and without in-degree and out-degree correlations (Fig. 2*D*) show similar behaviors as p varying from 1 to 0 such that

i) when the coupling strength $q < q_{c2}$ (q_{c2} is one critical coupling strength), $p_\infty^{(s)}$ continuously decreases to zero at a

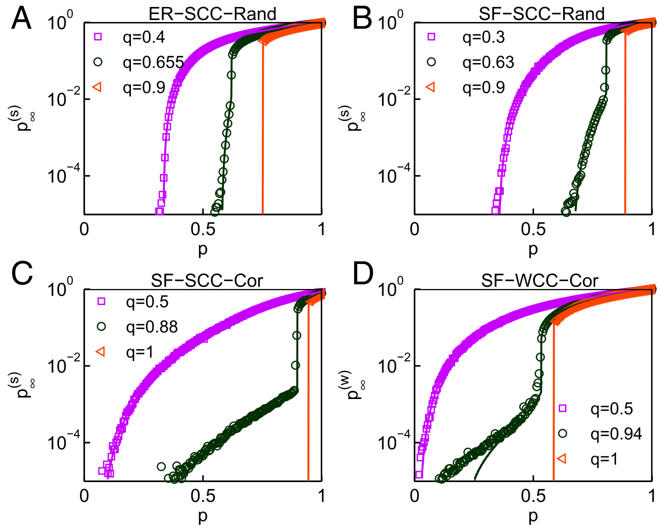

Fig. 2. Percolation of interdependent directed ER networks and SF networks with and without degree correlations. (*A*) The final GSCC size of interdependent directed ER networks ($\langle k\rangle = 10$) as a function of p for three different values of coupling strength q. The system shows a second-order phase transition when $q = 0.4 < q_{c2} = 0.646$, a hybrid phase transition when $q_{c2} < q = 0.655 < q_{c1} = 0.692$, and a first-order phase transition when $q = 0.9 > q_{c1}$. (*B*) The final GSCC size of interdependent directed SF networks ($\lambda = 2.8$) without in-degree and out-degree correlations with critical coupling strengths $q_{c2} = 0.597$ and $q_{c1} = 0.707$. (*C*) The final GSCC size of interdependent directed SF networks with in-degree and out-degree correlations with $q_{c2} = 0.823$ and $q_{c1} = 0.972$. (*D*) The final GWCC size of interdependent directed SF networks with in-degree and out-degree correlations. The simulation results (averaged over 60 runs and $N = 10^6$) are shown using symbols and the theoretical predictions are obtained by using Eq. **1** and substituting the corresponding generating functions of ER networks (*SI Appendix*, Eq. **S26**) and SF networks with and without (*SI Appendix*, Eqs. **S39** and **S45**) in-degree and out-degree correlations, respectively.

percolation threshold p_c^{II}, characterized as a second-order phase transition;

ii) when $q > q_{c1}$ (q_{c1} is another critical coupling strength), $p_\infty^{(s)}$ discontinuously jumps to zero at another percolation threshold p_c^I, characterized as a first-order phase transition; and

iii) when $q_{c2} < q < q_{c1}$, $p_\infty^{(s)}$ sharply jumps to zero at p_c^I followed by a second-order phase transition at p_c^{II}, characterized as a hybrid phase transition (17).

Note that in the interdependent undirected ER networks, there is no hybrid phase transition (19).

The percolation behaviors above can be analytically computed using the function $R^{(s)}(z,q)$. When the system exhibits a second-order phase transition, as shown in Fig. 3*A*, $R^{(s)}(z,q)$ is a monotonically increasing function of z, and the maximum value of $R^{(s)}(z,q)$ is obtained when $z \to 1$. In addition, for the hybrid phase (Fig. 3*B*), $R^{(s)}(z,q)$ is a nonmonotonic increasing function of z, but the maximum value of $R^{(s)}(z,q)$ is still obtained when $z \to 1$, corresponding to the reciprocal of percolation threshold p_c^{II}. Thus, the percolation threshold p_c^{II} can be written as

$$p_c^{II} = \frac{1}{\lim_{z\to 1} R^{(s)}(z,q)} = \frac{1}{\Phi_1'(1,1)(1-q)}. \qquad \textbf{[2]}$$

Note that for the case of ER networks with average degree $\langle k\rangle$, we obtain $p_c^{II} = 2/(\langle k\rangle(1-q))$. As shown in both Fig. 3 *B* and *C*, when the system displays a hybrid or a first-order phase transition, $R^{(s)}(z,q)$ as a function of z has a peak at z_c, and z_c is the smaller root of $\partial_z R^{(s)}(z,q) = 0$. Accordingly, the percolation threshold p_c^I is

$$p_c^I = \frac{1}{R^{(s)}(z_c,q)}. \qquad \textbf{[3]}$$

We apply our theoretical prediction of the percolation thresholds to interdependent directed ER networks and SF networks. Fig. 3 *D*–*G* shows that the percolation threshold p_c^{II} (solid lines) increases gradually for $q \in [0, q_{c1}]$ and that the other percolation threshold p_c^I (dashed line) appears at $q = q_{c2}$ and then increases for $q \in [q_{c2}, 1]$. In the region $[q_{c2}, q_{c1}]$, two percolation thresholds p_c^{II} and p_c^I coexist, indicating a hybrid phase transition. The percolation thresholds p_c^{II} and p_c^I in interdependent directed SF networks (Fig. 3*F*) are larger than those in interdependent undirected networks (Fig. 3*G*), because nodes in the GSCC must reach and be reached by one another and nodes in the GWCC must reach or be reached by one another.

One critical coupling strength q_{c2} separates the second-order and the hybrid phase transitions, and another critical coupling strength q_{c1} separates the hybrid and the first-order phase transitions. These two critical coupling strengths can be analytically

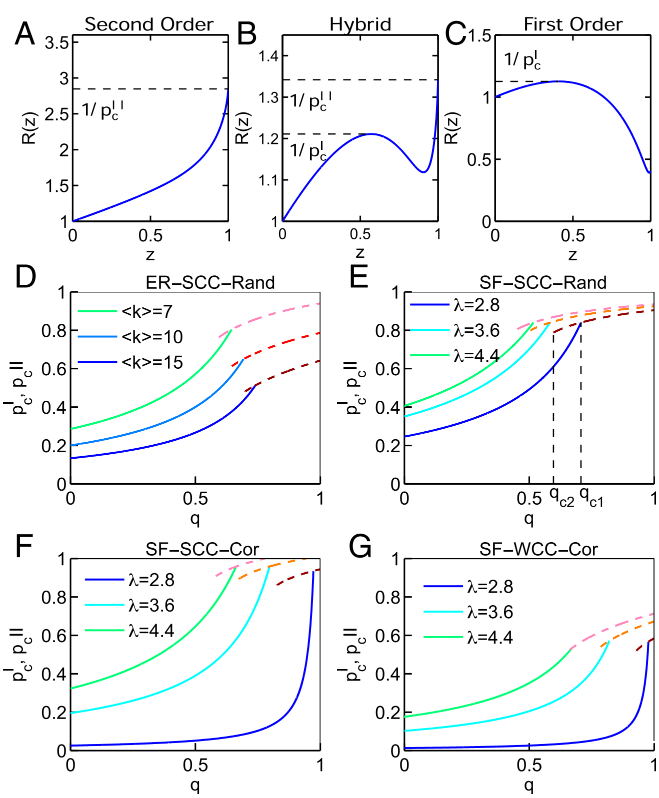

Fig. 3. Percolation thresholds of interdependent directed ER networks and SF networks with and without degree correlations. (*A*) When the coupling strength is $q \le q_{c2}$, $R^{(s)}(z,q)$ is an increasing function of z, revealing a second-order phase transition with a percolation threshold p_c^{II}. (*B*) When the coupling strength is $q_{c2} \le q \le q_{c1}$, $R^{(s)}(z,q)$ shows a maxima as $1/p_c^I$ and its maximum value is obtained when $z \to 1$ as $1/p_c^{II}$, indicating a hybrid phase transition with two percolation thresholds, p_c^{II} and p_c^I. (*C*) When the coupling strength is $q \ge q_{c1}$, the maxima of $R^{(s)}(z,q)$ is also the maximum for $z \in [0,1]$, exhibiting a first-order phase transition with a percolation threshold p_c^I. In (*D*) interdependent directed ER networks, (*E*) SF networks without in-degree and out-degree correlations, and (*F* and *G*) SF networks with in-degree and out-degree correlations, the percolation threshold p_c^{II} (solid lines) increases gradually for $q \in [0, q_{c1}]$, and the other percolation threshold p_c^I (dashed lines) appears at $q = q_{c2}$ and then increases for $q \in [q_{c2}, 1]$. In the region $[q_{c2}, q_{c1}]$, two percolation thresholds p_c^{II} and p_c^I coexist, indicating a hybrid phase transition. The percolation thresholds p_c^{II} and p_c^I in (*F*) interdependent directed SF networks are larger than those in (*G*) interdependent undirected networks.

solved using the function $R^{(s)}(z,q)$ and its derivations. We draw the phase diagram for interdependent directed ER networks in the $q-\langle k\rangle$ plane and the phase diagrams for interdependent SF networks in the $q-\lambda$ plane, as shown in Fig. 4. We observe a hybrid phase transition in interdependent directed ER networks, as shown in Fig. 4*A*, which is not observed in interdependent undirected ER networks (19). The region of the first-order and the hybrid phase transitions in interdependent directed SF networks (Fig. 4*C*) is much broader than that in interdependent undirected SF networks (Fig. 4*D*). The critical coupling strengths q_{c2} and q_{c1} of interdependent SF networks without in-degree and out-degree correlations (Fig. 4*B*) are, respectively, smaller than those of interdependent SF networks with in-degree and out-degree correlations (Fig. 4*C*), indicating that the in-degree and out-degree correlations dramatically enlarge the region of second-order phases and shrink the region of first-order phases. If the degree distribution exponent $2\le\lambda\le3$ and the scales of both networks $N\to\infty$, then $q_{c1}=1$ in interdependent directed SF networks with in-degree and out-degree correlations (Fig. 4*C*) and undirected SF networks (Fig. 4*D*), i.e., the first-order phase transition, emerges only if $q=1$ (*SI Appendix*, section VI). When interdependent directed SF networks have no in-degree and out-degree correlations, as shown in Fig. 4*B*, $q_{c1}<1$ even when $2\le\lambda\le3$, indicating that they are much more vulnerable than when correlations are present.

Influence of In-Degree and Out-Degree Correlations

The robustness of the interdependent networks can be defined either by the percolation thresholds denoted by R^{I}, where the smaller the value of R^{I} the more robust the system (criterion I), or by the integrated size of the final GSCC during the entire attack process, denoted by $R^{II}=\int_0^1 p_\infty^{(s)}(p)dp$ (20), where the larger the value of R^{II} the more robust the system (criterion II). In previous research on the robustness of interdependent undirected networks, these two criteria always led to the same result when comparing the robustness of two systems with different average degrees (15), coupling strengths (19), degree exponents (17), and even with a different number of coupled networks (20, 22). However, when we compare the robustness of two interdependent directed SF networks with and without in-degree and out-degree correlations, opposite results can appear (the purple region of Fig. 5) when using these two criteria for some certain degree exponents and coupling strengths. This indicates that either of these two criteria used alone is insufficient for characterizing the robustness of interdependent networks, so we must use both.

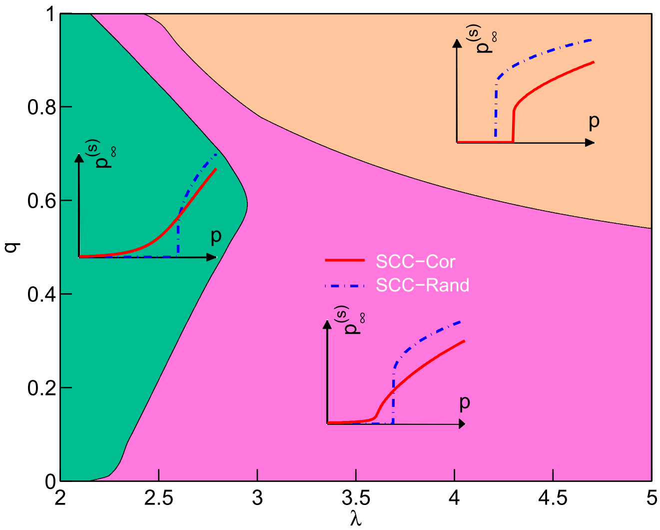

Fig. 5. Influence of the in-degree and out-degree correlations on robustness. We compare the robustness of interdependent SF networks with and without in-degree and out-degree correlations using criteria I and II. We divide the $q-\lambda$ space into three regions. In each region we show the illustrative percolation curves of the two systems. The robustness of the correlated SF networks is higher than when there is no correlation (green region where $R^{I}_{\mathrm{Cor}}>R^{I}_{\mathrm{Rand}}$ and $R^{II}_{\mathrm{Cor}}>R^{II}_{\mathrm{Rand}}$), and lower in the orange region where $R^{I}_{\mathrm{Rand}}>R^{I}_{\mathrm{Cor}}$ and $R^{II}_{\mathrm{Rand}}>R^{II}_{\mathrm{Cor}}$, as determined using criteria I and II. In the purple region where $R^{I}_{\mathrm{Cor}}>R^{I}_{\mathrm{Rand}}$ and $R^{II}_{\mathrm{Rand}}>R^{II}_{\mathrm{Cor}}$, the robustness of the correlated SF network is higher than the robustness of the uncorrelated SF networks when using criterion I, and lower when using criterion II.

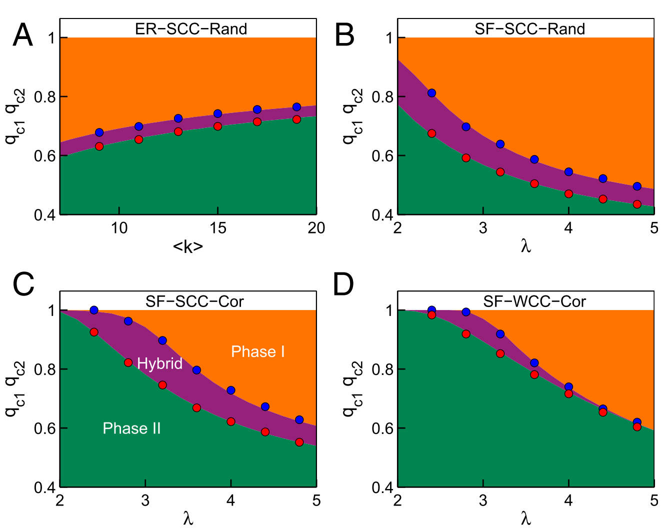

Fig. 4. Phase diagrams of interdependent directed ER networks and SF networks with and without degree correlations. Systems exhibit different phase transitions when the coupling strength q varies: a second-order ($q<q_{c2}$, green region labeled "Phase II"), a first-order ($q>q_{c1}$, orange region labeled "Phase I"), and a hybrid phase transition ($q_{c2}<q<q_{c1}$, purple region labeled "Hybrid"). (*A*) A hybrid phase transition emerges in interdependent directed ER networks that does not occur in coupled undirected ER networks (19). The critical coupling strengths q_{c2} and q_{c1} in interdependent SF networks (*B*) without in-degree and out-degree correlations are smaller than (*C*) when no degree correlations exist. This indicates that the in- and out-degree correlations increase the robustness. The region of the first-order and hybrid phase transitions in the percolation of interdependent directed SF networks (*C*) is much broader than the region in interdependent undirected SF networks (*D*). This indicates that directionality increases the system vulnerability. The lines between phases are predicted by our analytic framework, and the symbols are simulation results ($N=10^6$ nodes in each network).

We compare the robustness of two systems of SF networks with and without in-degree and out-degree correlations in the $\lambda-q$ plane. As shown in Fig. 5, in the yellow and green regions, either system shows a higher or lower robustness in both R^{I} and R^{II}, whereas in the purple region, one system shows higher robustness in R^{I} and lower robustness in R^{II}. Moreover, Fig. 5 shows that when using criterion I, in-degree and out-degree correlations increase the robustness of the system of networks with a degree exponent $2<\lambda<3$, called heterogeneous networks, while decreasing the robustness of the system of networks with $\lambda>3$, called homogeneous networks, with a strong coupling strength q (the orange area). When using criterion II, the robustness of systems composed of networks with degree exponents $2<\lambda<3$ also increases with the in-degree and out-degree correlations (the green area). Thus, the in-degree and out-degree correlations increase the robustness of interdependent directed networks with heterogeneous degree distributions but decrease the robustness of interdependent networks with homogeneous degree distributions and with strong coupling strengths. Most networks in the real world are heterogeneous, so the in-degree and out-degree correlations can increase the robustness of real interdependent directed networks.

Application to Real Networks

We use eight commodity-specific trade networks among world countries in 2013 (34), a list of which is displayed in *SI Appendix*, Table S1. In each trade network consisting of $N = 240$ countries, a directed link from node i to node j indicates that country i sells the commodity to country j. In multiplex networks two nodes in different layers of the network depend on each other because they represent the same individual or share the same property in different layers (35), thus in each pair of interdependent trade networks two nodes in different networks depend on each other because they represent the same country. In addition, the percolation modeling that we apply is related to the study of percolation in multiplex networks in the context of interdependent networks (35). Using the partially interdependent undirected network in ref. 19, we assume that a fraction of q nodes in each network depends on nodes in other networks and forms a network of eight commodity-specific trade networks in which each pair of networks forms a subsystem of interdependent networks connected by q fraction of dependency links, and the total number of subsystems is 28.

Fig. 6*A* shows the robustness R^{II} of these eight single networks averaged over 600 attack events, which indicates that R^{II} is a good measure of the robustness because the error bars are small. Fig. 6 *B* and *C* relates the size of a node to the robustness R^{II} of an individual network. The width of the links is related to the R^{II} of the network with higher robustness (Fig. 6*B*) and the network with lower robustness (Fig. 6*C*) in the system, indicating that the robustness of this system is primarily determined by the lower robustness level. Apart from a few subsystems whose simulation values of robustness are slightly higher than the theory computing values due to the degree–degree correlations between networks (*SI Appendix*, Fig. S4), the robustness of most subsystems calculated by our theory framework R^{II}_{Theo} agrees well with the simulation results R^{II}_{Sim} (Fig. 6*D*), and the degree–degree correlations between networks is caused by the fact that a hub country in one network (trading soaps) may also be a hub in another network (trading papers). Fig. 6*E* shows that the ER randomized networks (*SI Appendix*, section VIII) are much more robust than those in real-world interdependent networks because, for a certain commodity, there are many countries that only sell or only buy it. According to definition, every node within a GSCC has outgoing links that reach to other nodes and incoming links that reach from other nodes, thus a country that only sells a certain commodity (no outgoing links from this node) or only buys the commodity (no incoming links to this node) does not belong to the GSCC, and in real interdependent networks this dramatically decreases the final GSCC size. If we preserve the elements of the in-degree and out-degree sequences and then randomize the networks (*SI Appendix*, section VIII), these randomized networks are less robust than those in real-world systems, shown in Fig. 6*F*, because the in-degree and out-degree correlations in the interdependent real-world networks increase their robustness.

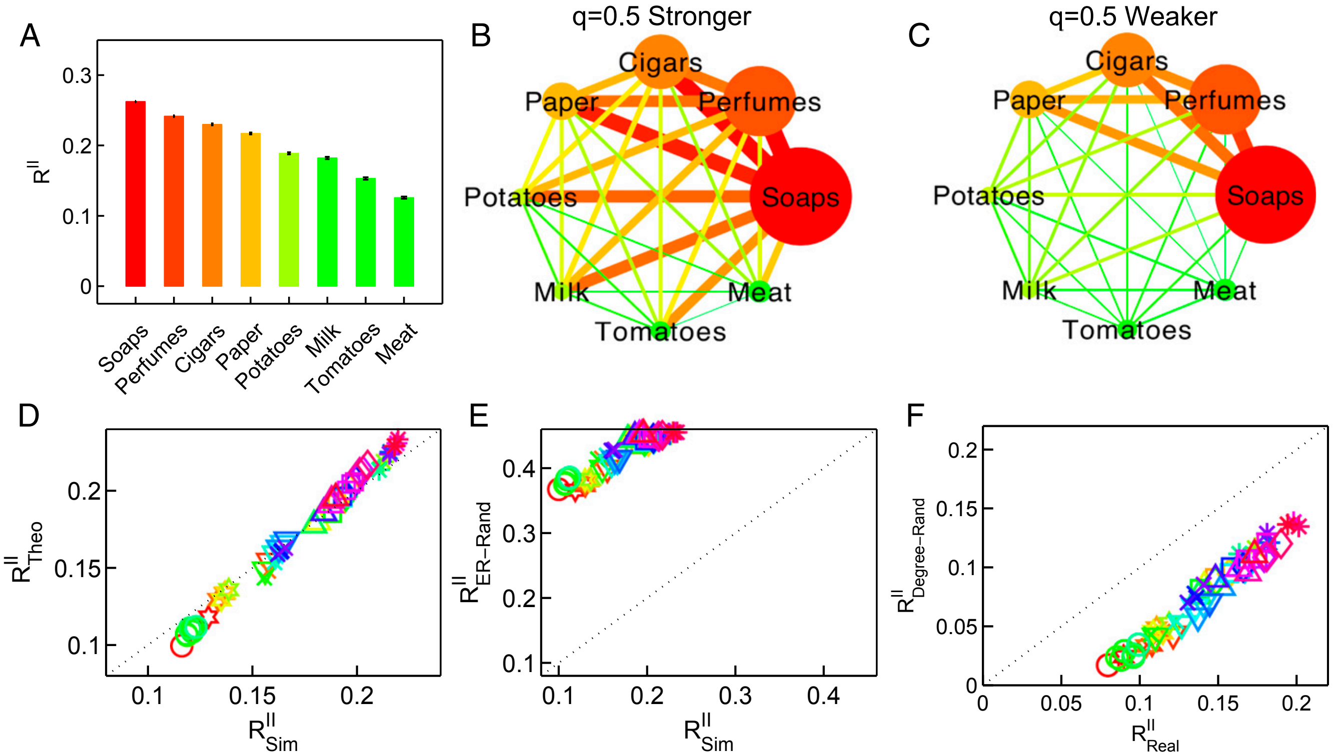

Fig. 6. Robustness of interdependent international trade networks. (*A*) The robustness R^{II} of eight individual international trade networks that constitute a network of trade networks in which each pair of networks forms a subsystem of interdependent networks connected by q fraction of dependency links (here we set $q = 0.5$ as an example). The size of a node is related to the R^{II} of an isolated network, and the width of the links is related to the (*B*) higher R^{II} and (*C*) lower R^{II} of two coupled networks. (*B*) Each link of the same node is narrower when it connects to a smaller node, indicating that the robustness of a system is determined by the smaller of the two networks. (*D*) The robustness of 28 subsystems of interdependent international trade networks R^{II}_{Theo}, calculated by our theory framework, agrees with simulation results R^{II}_{Sim} except when $R^{II}_{\mathrm{Sim}} > R^{II}_{\mathrm{Theo}}$, which is caused by degree–degree correlations between networks. (*E*) ER randomized networks are more robust than real networks $R^{II}_{\mathrm{ER-Rand}} > R^{II}_{\mathrm{Sim}}$ because many participating countries only sell or only buy one commodity. (*F*) If we preserve the elements of the in-degree and out-degree sequences and randomize the networks, these randomized networks are less robust than real networks $R^{II}_{\mathrm{Sim}} > R^{II}_{\mathrm{Degree-Rand}}$ because the in-degree and out-degree correlations in interdependent real networks increase their robustness. Each result is averaged over 600 realizations.

Discussion

In summary, we have developed a general theoretical framework for analyzing the breakdown of interdependent directed networks and have discovered that the behavior of interdependent directed networks differs from that in interdependent undirected networks. For example, the interdependent directed ER networks system shows a hybrid phase transition that is not observed in coupled undirected ER networks (19). We find that the in-degree and out-degree correlations in each layer of network could increase the robustness of interdependent networks with heterogeneous degree distributions, but decrease the robustness of interdependent networks with homogeneous degree distributions and with strong coupling strengths. By comparing the robustness of real-world interdependent international trade networks with the robustness of the in-degree and out-degree distribution that preserves randomized networks, we find that the in-degree and out-degree correlations improve the robustness of real-world interdependent networks.

Real-world interdependent international trade networks are less robust than ER randomized networks because some countries in the network only sell or only buy a particular commodity and thus do not belong to the GSCC. To increase the robustness of the system, countries that only sell a certain commodity must be encouraged to buy the same commodity, and vice versa. Note that this "encouragement" harmonizes with the structural properties of the socioeconomic dynamics driving trade networks in which the nodes are connected by bilateral links, and that these socioeconomic dynamics driving trade networks are usually robust (36). The robustness of interdependent trade networks can also be increased by increasing the in-degree and out-degree correlations between the participating countries. Moreover, in our work we study the robustness of the subsystems composed by two of eight trade networks. When we treat the eight together as a multiplex network with eight fully interdependent layers, it will be much less robust than any of its subsystem, and its final GSCC size can be solved by substituting *SI Appendix*, Eq. **S5** into equation 30 in ref. 37, which could trigger some open questions for future studies: (i) What is the theoretical framework of the robustness of a network of more than two directed networks? (ii) How do the degree–degree correlations between networks influence the robustness of interdependent directed networks?

Methods

Generating Functions. Given a directed network with a degree distribution $P(k_{\text{in}}, k_{\text{out}})$, whose generating function is $\Phi(x,y)=\sum_{k_{\text{in}},k_{\text{out}}}^{\infty} P(k_{\text{in}}, k_{\text{out}})x^{k_{\text{in}}}y^{k_{\text{out}}}$ (28), $\Phi_1(x,1)=\partial_y\Phi(x,y)\big|_{y=1}/\partial_y\Phi(1,1)$ and $\Phi_1(1,y)=\partial_x\Phi(x,y)\big|_{x=1}/\partial_x\Phi(1,1)$ are the generating functions of branching processes.

Percolation Theory. The size of the GSCC for a single network is $S=1-\Phi(x_c,1)-\Phi(1,y_c)+\Phi(x_c,y_c)$, where $x_c=\Phi_1(x_c,1)$ and $y_c=\Phi_1(1,y_c)$. We propose a general analytic framework to study the breakdown process in interdependent directed networks based on generating functions and percolation theory. The *SI Appendix*, section III contains further information.

In-Degree and Out-Degree Correlations. In some real-world networks, the in-degree k_{in} and out-degree k_{out} of a given node are correlated with the form $k_{\text{out}} \propto k_{\text{in}}^{\alpha}$ (32, 33). The correlation between in-degree and out-degree is positive, negative, or not correlated when the coefficient $\alpha \in (0,1]$, $\alpha \in [-1,0)$ and $\alpha=0$, respectively. For simplicity, we set $\alpha=1$ in our numerical simulations, but the extension to other values of α is straightforward.

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