SOME RIGOROUS RESULTS CONCERNING THE CROSSOVER BEHAVIOR
OF THE ISING MODEL WITH LATTICE ANISOTROPY *

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The following rigorous relations are established for the Ising model with interaction strengths $J$ in some lattice directions and $RJ$ in other directions: $\gamma_1 = 2\gamma, \gamma_2 \geq 3\gamma,$ and $\gamma_3 \geq 4\gamma,$ where $\chi_n (0) = (\delta^n \chi / \delta R^n)_{R=0} \sim e^{-\gamma_n}$, and $\gamma_0 = \gamma$ is the susceptibility exponent for the lattice when $R=0$. These results disagree with recently-reported numerical estimates of certain of the $\gamma_n$.

There has recently been considerable interest in systems with “lattice anisotropy” (different coupling strengths in different lattice directions). Consider, e.g., the $d$-dimensional nearest-neighbor (nn) Ising system

$$\mathcal{H} = -J \sum_{i=1}^{\text{nn}} s_i s_j - R J \sum_{i=1}^{\text{nn}} s_i s_j,$$

$$\equiv \mathcal{H}_0 + R \mathcal{H}_1,$$ (1)

where $r_i = (x_1, x_2, \ldots, x_d)$ and $u_i = (x_1, \ldots, x_d)$. For example, very recently there have been extensive calculations [1, 2] concerning the case $d=3, d=2$, corresponding to a “square to simple cubic crossover”. Henceforth we shall consider this system for the purpose of specificity and clarity; thus $r_i = (x_i, \psi_i, z_i) = (u_i, z_i)$ and $R \equiv J_x / J_{xy}$. Our approach is, however, more general.

According to the generalized scaling hypothesis, for which the parameter $R$ is scaled (as well as $\varepsilon, H, \ldots$), the “crossover” exponent $\phi$ is the only exponent that one needs to describe the crossover behavior [8]. In particular,

$$\gamma_n = \gamma + n \phi,$$ (2)

where the new exponent $\gamma_n$ is defined by

$$\chi_n (R=0) \equiv (\delta^n \chi / \delta R^n)_{R=0} \sim [T - T_c (0)]^{-\gamma_n}.$$ (3)

Here $\chi$ is the reduced zero-field magnetic susceptibility and $\gamma_0 = \gamma$ is the susceptibility exponent of the $d$-dimensional system.

The exponents $\gamma_n$ cannot be calculated exactly but they can be estimated by extrapolations based upon high-temperature series expansions. There presently exists a dispute [1, 3 – 5] in the literature concerning numerical values of $\gamma_n$, and the most recent work claims that for sq $\rightarrow$ sc Ising model,

$$\gamma_1 = 3.5, \quad \gamma_2 = 5.0 \pm 0.1, \quad \gamma_3 = 6.5 \pm 0.2, \quad \gamma_4 = 8.0 \pm 0.3.$$ (4)

In this note we shall report the following rigorous results:

$$\gamma_1 = 2\gamma$$ (5a)

$$\gamma_2 \geq 3\gamma$$ (5b)

$$\gamma_3 \geq 4\gamma.$$ (5c)

Since $\gamma = 1.75$ for a sq Ising model, the numerical estimates of (4) violate (5). Our results also lend support for the predictions (2) and $\gamma_n = (n + 1) \gamma$.

As a demonstration, we shall here outline the proof of (5b). Details of the analysis will be published elsewhere.

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The expressions (7a) — (7d) are weighted by factors $4(N-1)$, $N(N-1)$, $2N$, and $2N$ respectively, arising from the fact that we can make interchanges of the form $i \leftrightarrow j$ etc. in fig. 1.

The second term in (6) has two factors,

$$
\sum R_i R_j (s_i s_j) = (N+1)M^2 x_0 (0)
$$

and

$$
\langle \mathcal{H}_i^2 \rangle = J^2 NM^2 \sum_u \langle s_0 s_u \rangle^2.
$$

Thus (6) becomes

$$
(\beta J)^{-2} (N+1)M^2 x_2 (0) = 4(N-1)M^2 [x_0 (0)]^3
$$

$$
-2NM^2 x_0 (0)M^2 \sum \langle s_0 s_u \rangle^2
$$

$$
+2NM^2 \sum \langle \sum_i s_i \rangle^2 \langle s_0 s_u \rangle \langle s_0 s_u \rangle
$$

$$
+2NM^2 \sum \langle s_0 s_u \langle \sum_i s_i \rangle^2.
$$

The Griffiths inequality [11],

$$
\langle s_i s_j s_k s_l \rangle \geq \langle s_0 s_u \rangle \langle s_0 s_u \rangle
$$

permits us to "cancel" the second and third terms on the right-hand side of eq. (10), and noting that the fourth term is positive, we have

$$
\chi_2 (0) \geq 4(\beta J)^2 [x_0 (0)]^3
$$

where we have neglected $O(1/N)$ with respect to unity, inequality (5b) follows from (11).

In conclusion, we have shown rigorously that $\gamma_1 = 2\gamma$, $\gamma_2 \geq 3\gamma$, and $\gamma_3 \geq 4\gamma$. If the scaling hypothesis is valid (so that $\gamma_n = \gamma + n\delta$), our work furnishes a simple but rigorous proof of $\phi = \gamma$. Moreover, our results (5b) and (5c) indicate that reported values of $\gamma_2$ and $\gamma_3$ are unreliable [1—3]. A detailed study of these (and other [12]) high-temperature series for the lattice anisotropy problem is now underway, and preliminary
numerical results indicate that $\gamma_n = (n+1)\gamma$ for $n = 1, 2, 3, 4$.

References