times in a superconductor such as electron-electron, electron-phonon, and branch mixing[10] must be considerably shorter. In addition, the behavior of the energy gap of an illuminated superconductor can be used to measure the quasiparticle recombination time in superconductors.

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Some Results Concerning the Crossover Behavior of Quasi–Two-Dimensional and Quasi–One-Dimensional Systems*

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A magnetic system with intraplanar and interplanar interaction strengths J and RJ is is treated. Rigorous relations are established concerning the first few derivatives with respect to R of the susceptibility $\chi(R)$. Considering $\chi(R)=b_0+b_1R+b_2R^2+\cdots$, we find b_1 and the order of magnitude of b_2. Hence we can predict when the system "crosses over" from d-dimensional to $\bar{d}$-dimensional behavior (e.g., for quasi–two-dimensional systems, $d=2$, $\bar{d}=3$, while for quasi–one-dimensional systems, $d=1$, $\bar{d}=3$). These results also support scaling with respect to the anisotropy parameter R.

There has recently been considerable interest[1-3] in systems with "lattice anisotropy" (different coupling strengths in different lattice directions). Consider, e.g., the d-dimensional nearest-neighbor (nn) Ising system with Hamiltonian

$$\mathcal{H}=-J\sum_{\vec{v}_i=\vec{v}_j}^{\text{nn}} s_i s_j - RJ\sum_{\vec{u}_i=\vec{u}_j}^{\text{nn}} s_i s_j \equiv \mathcal{H}_0+R\mathcal{H}_1, \qquad (1)$$

where $\vec{r}_i\equiv(x_1,x_2,\cdots,x_{\bar{d}})\equiv(\vec{u}_i,\vec{v}_i)$ with $\vec{u}_i=(x_1,\cdots,x_d)$, and $\vec{v}_i\equiv(x_{d+1},\cdots,x_{\bar{d}})$. For example, very recently there have been extensive calculations[1] concerning the case $d=2$, $\bar{d}=3$, corresponding to a "square to simple-cubic crossover." Henceforth we shall consider this system for the purpose of specificity and clarity; thus $\vec{r}_i\equiv(x_i,y_i,z_i)\equiv(\vec{u}_i,z_i)$, $J=J_{xy}$, $RJ=J_z$. In the last paragraph we treat briefly the case $d=1$, $\bar{d}=3$.

The system described by (1) is interesting because critical-point exponents, according to the universality hypothesis,[2] should depend only upon lattice dimensionality; and hence when $R\to 0$ (and the lattice "crosses over" from $\bar{d}$ dimensions), we expect anomalous behavior. This crossover behavior would be observable *if* we could vary R continuously to zero.

Another interesting property of the weakly coupled layers is that even for $R\neq 0$ the system is essentially two-dimensional at high temperature. Yet when it is sufficiently close to the critical temperature $T_c(R)$, it is *three*-dimensional. Hence there is a crossover region $T_A(R)\geq T\geq T_B(R)$ where the system transits from $d=2$ to $\bar{d}=3$ (cf. Fig. 1).

The crossover region is only a loosely defined concept. To be quantitatively precise, we shall define $T_p(R)$ as the solution of $\chi(T_p,R)/\chi(T_p,0)-1=p$ and we arbitrarily choose $T_A(R)=T_{0.01}(R)$, since this is roughly the temperature for which the deviation (1%) of the reduced susceptibility $\bar{\chi}(R)\equiv\chi/\chi_{\text{Curie}}$ from the two-dimensional value $\bar{\chi}(0)$ becomes experimentally appreciable.[4]

This crossover behavior is most easily explained in the context of the scaling hypothesis,[2] where we assume that there exist three numbers

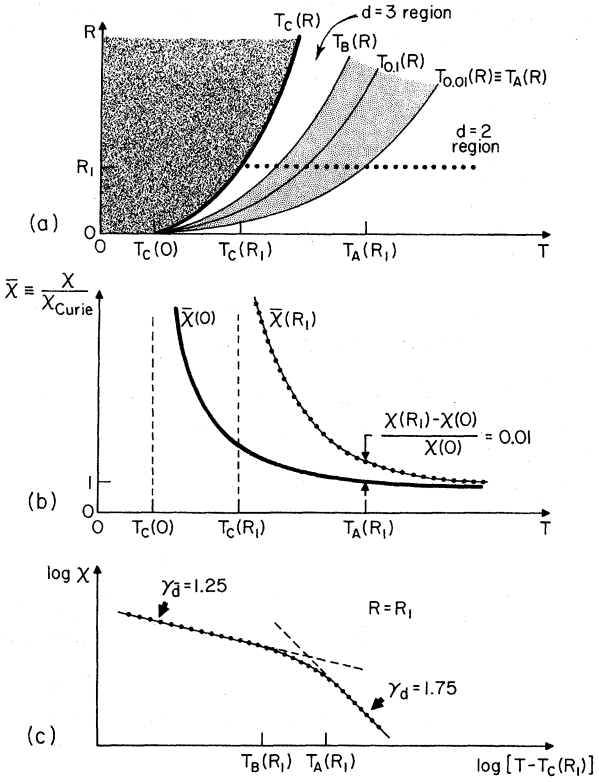

FIG. 1. Schematic diagram of the crossover behavior. (a) The crossover region (shaded area) is bounded by $T_A(R)$ and $T_B(R)$. [$T_A(R) = T_{0.01}(R)$ is the temperature at which the system differs appreciably (1%) from being two dimensional.] $T_{0.1}(R)$ is the temperature at which Eq. (4) is no longer satisfactory. $T_c(R)$ is the critical temperature. The generalized scaling hypothesis predicts that all curves should approach $T_c(0)$ via the power law $R^{1/\varphi}$. (b) Dependence of reduced susceptibility $\bar{\chi}$ upon T for $R=0$ and for $R=R_1$, indicating the definition of $T_A(R)$. Note that this drawing is not to scale. (c) Sketch of hypothetical experimental data, plotted in the conventional log-log plot, for a system which is described by the Hamiltonian (1) with $R=R_1$.

a_τ, a_H, and a_R, such that for all positive λ

$$G(\lambda^{a_\tau}\tau, \lambda^{a_H}H, \lambda^{a_R}R) = \lambda G(\tau, H, R), \tag{2}$$

where G is the singular part of the Gibbs potential, $\tau = T - T_c$, and H is the magnetic field. From the scaling hypothesis, the situation sketched in Fig. 1 follows immediately.

In particular, we note that (2) implies that $T_c(R) - T_c(0) \sim CR^{1/\varphi}$ and $T_p(R) - T_c(0) \sim A_p R^{1/\varphi}$, where $\varphi \equiv a_R/a_\tau$ is the "crossover exponent." On differentiating (2) twice with respect to H and n times with respect to R, we have $\bar{\chi}_n(\tau, H=0, R=0) \sim |\tau|^{-\gamma_n}$, with $\gamma_n = \gamma_0 + n\varphi$, where $\bar{\chi}_n \equiv (\partial^n \bar{\chi}/dR^n)_{H,\tau}$.

The exponents γ_n cannot be calculated exactly, but they *can* be estimated by extrapolations based upon high-temperature series expansions. There presently exists a dispute[1b,1c] in the literature concerning numerical values of γ_n, and the most recent estimates[1b] seem to challenge the scaling prediction $\gamma_n = \gamma_0 + n\varphi$.

We shall first derive rigorously the relation

$$\bar{\chi}_1(0) = 2g\bar{\chi}_0^2, \tag{3}$$

where $g \equiv \beta J$. This relation is useful for the following purposes:

(i) Equation (3) is the first-order correction term of the two-dimensional approximation of the three-dimensional quantity $\bar{\chi}(R)$. In other words, we have

$$\bar{\chi}(R) = \bar{\chi}_0 + R\bar{\chi}_1(0) + \tfrac{1}{2}R^2\bar{\chi}_2(0) + \cdots = \bar{\chi}_0 + R(2g\bar{\chi}_0^2) + O(R^2). \tag{4}$$

To $O(R)$, $T_p(R)$ may therefore be found from the solution of $2gR\bar{\chi}_0 = p$.

(ii) Equation (4) may be used as a checking method for the leading coefficient of the general-R high-temperature series expansion

$$\bar{\chi}(R) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_{nj}R^j g^n. \tag{5}$$

(iii) If (2) is valid, Eq. (3), which yields $\gamma_1 = 2\gamma_0$, furnishes a simple *but rigorous* proof of $\varphi = \gamma_0$, and hence $\gamma_n = (n+1)\gamma_0$.

We shall now derive the following results:

$$8g^2\chi_0^3 \geq \bar{\chi}_2(0) \geq 4g^2\bar{\chi}_0^3, \tag{6}$$

$$48g^3\chi_0^4 \geq \bar{\chi}_3(0) \geq 8g^3\bar{\chi}_0^4. \tag{7}$$

Equations (6) and (7) yield $\gamma_2 = 3\gamma_0$, and $\gamma_3 = 4\gamma_0$, which agree exactly with the scaling predictions $\gamma_n = (n+1)\gamma_0$; thus the reported estimates[1b] are unreliable. From (6) we have a fairly good estimate of

the second-order correction term; therefore (4) may be improved, with the result

$$\bar{\chi}(R) \cong \bar{\chi}_0[1 + 2R\mathcal{J}\bar{\chi}_0 + R^2 4\mathcal{J}^2\bar{\chi}_0^2 f_2(\mathcal{J}) + O(R^3)], \tag{8}$$

where $2 \geq f_2(\mathcal{J}) \geq 1$.

The proof of (3) proceeds from observing that at $R=0$, spins on different layers (i.e., if $z_k \neq z_l$) are not coupled. Hence for $T > T_c(0)$ and $H=0$ (i.e., $\langle s_i \rangle_d = 0$), the thermal average of spin product is

$$\langle s_k s_l \rangle_{R=0} = \delta(z_k, z_l)\langle s_k s_l \rangle_d, \tag{9}$$

where $\langle \cdots \rangle_R$ denotes a thermal average of a $\bar{d}$-dimensional Hamiltonian (1) and $\langle \cdots \rangle_d$ denotes a thermal average of a d-dimensional Hamiltonian. Note that on the right-hand side of Eq. (9) we set $R=0$ in the Hamiltonian *before* taking a thermal average, while on the left-hand side we set $R=0$ *after* calculating the thermal average. Furthermore we have, for $z_k \neq z_l$,

$$\langle s_i s_j s_k s_l \rangle_{R=0} = \delta(z_i, z_k)\,\delta(z_j, z_l)\langle s_i s_k \rangle_d \langle s_j s_l \rangle_d + \delta(z_i, z_l)\,\delta(z_j, z_k)\langle s_i s_l \rangle_d \langle s_j s_k \rangle_d. \tag{10}$$

The reduced susceptibility of a lattice which is composed of $N+1$ layers with M^2 spins in each layer[5] is given by

$$\bar{\chi}(T, H=0; R) = [(N+1)M^2]^{-1} \sum_{\vec{r}_i, \vec{r}_j} [\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle]. \tag{11}$$

Differentiating (11) and using (9) and (10), we have

$$(N+1)M^2\bar{\chi}_1(0) = -\beta \sum_{\vec{r}_i, \vec{r}_j} \langle s_i s_j \mathcal{H}_1 \rangle_{R=0} = \mathcal{J} \sum_{\vec{u}_k} \sum_{z_k=1}^{N} \Big\{ \sum_{\vec{u}_i} \langle s_{\vec{u}_i z_k} s_{\vec{u}_k z_k} \rangle_d \sum_{\vec{u}_j} \langle s_{\vec{u}_j z_k + 1} s_{\vec{u}_k z_k + 1} \rangle_d + \sum_{\vec{u}_i} \langle s_{\vec{u}_i z_k + 1} s_{\vec{u}_k z_k + 1} \rangle_d \sum_{\vec{u}_j} \langle s_{\vec{u}_j z_k} s_{\vec{u}_k z_k} \rangle_d \Big\}. \tag{12}$$

Now each of the four summations inside the curly brackets of (12) is over all spins which lie in a single plane, and thus each summation is exactly the reduced susceptibility of a *two*-dimensional lattice. Thus in the thermodynamic limit, we have Eq. (3).

Four remarks are worth making at this point:

(i) Equation (3) may be derived directly using graphical methods that are valid for arbitrary spin dimensionality D, and hence its validity is *not* restricted to the Ising model. In particular, for the spherical model ($D=\infty$), (3) can be derived directly from the general-R partition function as well.

(ii) The argument used in deriving (3) may be extended to any pair of lattices that can be related by a "lattice anisotropy" term in the Hamiltonian, with the result

$$\bar{\chi}_1^{\,d}(0) = g^{d\bar{d}}\mathcal{J}\bar{\chi}_0^{\,2}, \tag{13}$$

where $g^{d\bar{d}}$ is the number of "*additional* nearest neighbors" that a site acquires when the lattice anisotropy term transforms the lattice from being d-dimensional to being $\bar{d}$-dimensional [e.g., $g=2$ for sq $\rightarrow$ sc, linear chain $\rightarrow$ sq, but $g=4$ for linear chain (lc) $\rightarrow$ sc and $g=8$ for sq $\rightarrow$ fcc]. If $\mathcal{H}_1$ contains nonnearest-neighbor terms, we may simply replace $g^{d\bar{d}}\mathcal{J}$ by $\sum_i g_i^{\,d\bar{d}}\mathcal{J}_i$, where $g_i^{\,d\bar{d}}$ is the number of neighbors interacting with a site with strength RJ_i.

Equation (13) may be used as a checking procedure for the coefficients of (5) (e.g., at $R=0$, the a_{n1} are the coefficients in $2\chi_{\mathrm{sq}}^{\,2}$, while in the limit of $J \rightarrow 0$, $RJ = \mathrm{const}$, the $a_{n,n-1}$ are the coefficients in $4\bar{\chi}_{\mathrm{lc}}^{\,2}$).

(iii) Note that since our derivation starts from the correlation function, we may obtain the following relations for the second moment μ_2 of the correlation function:

$$(\partial\mu_2/\partial R)_{R=0} = 2\mathcal{J}[\bar{\chi}_0^{\,2} + 2\bar{\chi}_0\mu_2(0)], \tag{14}$$

and the general $\vec{q}$ susceptibility (where $\vec{q}$ is a $\bar{d}=3$-dimensional vector):

$$[\partial\bar{\chi}(\vec{q})/\partial R]_{R=0} = 2\mathcal{J}[\bar{\chi}_0(\vec{q})]^2 \cos(\vec{q}\cdot\hat{z}), \tag{15}$$

with $\bar{\chi}(\vec{q}) \equiv \int e^{i\vec{q}\cdot\vec{r}} C_2(\vec{r})\, d^3r$.

To derive (7) we may proceed in exactly the same fashion and arrive at

$$g^{-2}\bar{\chi}_2(0) = 4[\bar{\chi}_0(0)]^3 + 2\sum_{\vec{u},\vec{u}_i,\vec{u}_j} \langle s_0 s_{\vec{u}}\rangle_d (\langle s_0 s_{\vec{u}} s_{\vec{u}_i} s_{\vec{u}_j}\rangle_d - \langle s_0 s_{\vec{u}}\rangle_d \langle s_{\vec{u}_i} s_{\vec{u}_j}\rangle)_d + 2\sum_{\vec{u}} \langle s_0 s_{\vec{u}} (\sum s_{\vec{u}_i})\rangle_d{}^2. \quad (16)$$

The Griffiths inequality[6]

$$\langle s_0 s_{\vec{u}} s_{\vec{u}_i} s_{\vec{u}_j}\rangle - \langle s_0 s_{\vec{u}}\rangle \langle s_{\vec{u}_i} s_{\vec{u}_j}\rangle \geq 0 \quad (17)$$

permits us to cancel the second term on the right-hand side of (16). Noting that the third term is positive, we have the second inequality of (6). Using[7]

$$\langle s_0 s_{\vec{u}_i}\rangle \langle s_{\vec{u}} s_{\vec{u}_j}\rangle + \langle s_0 s_{\vec{u}_j}\rangle \langle s_{\vec{u}_i} s_{\vec{u}}\rangle \geq \langle s_0 s_{\vec{u}} s_{\vec{u}_i} s_{\vec{u}_j}\rangle - \langle s_0 s_{\vec{u}}\rangle \langle s_{\vec{u}_i} s_{\vec{u}_j}\rangle \quad (18)$$

and the fact that $\langle s_0 s_{\vec{u}} s_{\vec{u}_i}\rangle \equiv 0$ for the Ising system, we immediately have from (16) the first inequality of (6).

For higher derivatives, the algebra becomes much more involved. However, we have been able to derive (7) for systems for which Ginibre's generalization[8] of Griffiths's inequality[6] is valid. Furthermore, for the self-avoiding walk on an anisotropic lattice, by means of simple graphic theoretical arguments we have proved $\gamma_n = (n+1)\gamma_0$ for all n.

We conclude with several remarks. (i) The exponent φ obtained from the γ_n's does not necessarily characterize the asymptotic behavior of $T_c(R) - T_c(0)$ except within the framework of scaling-type arguments. (ii) The previous derivation[3] of $\varphi = \gamma_0$ exploits the relation $d\nu = 2 - \alpha$, for which numerical estimates indicate there is a 2% discrepancy for $d = 3$. Our derivation of (3) is rigorously true, *independent* of the relation $d\nu = 2 - \alpha$. (iii) It is clear that for different thermodynamic quantities, the crossover region may be different. In particular, since the first-order correction term $(\partial C_H/\partial R)$ in the analog of Eq. (4) for the specific heat is *zero*, it would be more difficult for one to use specific-heat measurements to cross over into the $d = 3$ region. (iv) One may also use (4) to determine the anisotropy parameter R. For example, experiments[9] show that $(CD_3)_4NMnCl_3$ is essentially an $S = \frac{5}{2}$ antiferromagnetic Heisenberg linear chain for $T \geq 1.1$°K with intrachain coupling $J = -6.3$°K. Assuming the experimental error bar is 5%, we may substitute $T = 1.1$°K, S, and J into the equation $4R\beta J\chi_{1c} \leq 0.05$; we get $R \lesssim 10^{-6}$, in contrast to the value $R \sim 10^{-5}$ obtained by a Green's function analysis.[9]

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