Analytic solution of the growth-site probability distribution for structural models of diffusion-limited aggregation

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We present an analytic solution of the growth-site probability distribution for a family of hierarchical models for the structure of diffusion-limited aggregation (DLA) clusters. These models are characterized by self-similar voids that are delineated by narrow channels. The growth-site probability distributions for all the models are shown to have the same form, $n(\alpha,M)\sim\exp\{-(A/\ln M)[\alpha-\alpha_0(M)]^2\}$, where $n(\alpha,M)d\alpha$ is the number of growth sites with $\alpha<-\ln p_i/\ln M<\alpha+d\alpha$, p_i is the growth probability at site i, M is the cluster mass, $\alpha_0(M)\equiv B\ln M$, and A,B are constants. We find the same form of the distribution for all members of the family of models, suggesting the possibility that it is a consequence of the channels and self-similar voids, and is independent of other details of the model. Our result is in accord with the recent calculations for DLA clusters by Schwarzer *et al.* [Phys. Rev. A **43**, 1134 (1991)].

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I. INTRODUCTION

Diffusion-limited aggregation [1] (DLA) has become important for describing a wealth of diverse physical, chemical, and biological phenomena [2]. Despite many ingenious attempts [2,3], no completely satisfactory understanding of DLA has emerged. One of the central problems is to understand how such a nontrivial pattern emerges from a simple set of rules for the growth. The dynamics of DLA, for a given cluster, can be described by the set of growth probabilities $\{p_i\}$, where p_i is the probability that site i will grow next [4].

There have been several attempts to measure and understand the distribution of p_i and its lower cutoff $p_{\min}$ [4–8]. However, the issue is still far from being settled. For example, the mass dependence of $p_{\min}$ is very controversial. There are suggestions of exponential [6], power-law [7], and an "intermediate" behavior [8(a)] $p_{\min}\sim\exp[-(\ln M)^2]$. Each of these forms can be related to a possible fjord structure. The exponential form [6] corresponds to narrow channels of length M^β with $\beta>0$, while the power-law form [7] corresponds to wedge-type fjords. The "intermediate" behavior [8] can be explained in terms of a structural model of DLA, which has self-similar voids connected by channels [8(a),9].

The scaling form of the growth-site probability distribution $\{p_i\}$ is also of interest. Trunfio and Alstrøm [6], Mandelbrot and Evertsz [6], and Schwarzer *et al.* [8(b)] proposed different types of possible behavior.

Here we shall propose a family of models designed to capture some of the essentials in the structure of DLA. These models, whose key ingredients are narrow channels and self-similar voids, are generalizations of the model presented in Refs. [8(a)] and [9]. We find an analytic solution of the growth-site probability distributions for the entire family of models. This distribution is found to have the same form as that of DLA clusters measured by Schwarzer *et al.* [8(b)], $n(\alpha,M)\sim\exp[-A(\ln M)^{-\delta}(\alpha-B\ln M)^\gamma]$. Here, $n(\alpha,M)d\alpha$ is the number of growth sites with $\alpha<-\ln p_i/\ln M<\alpha+d\alpha$. The two exponents ($\gamma=2\pm0.3,\delta=1.3\pm0.3$) used by Schwarzer *et al.* [8(b)] to characterize the distribution, are found for the present model to be 2 and 1, respectively. Furthermore, these exponents are *universal*, i.e., they are the same for the entire family of models. It is possible that the form of the distribution and the exponents are determined *only* by the presence of the channels and self-similar voids, independent of further details of models. The agreement between the distribution (and its exponents) of DLA and the models provides further support for the void-channel description of the structure of DLA.

The present paper is organized as follows. We define the family of models in Sec. II, and derive the recursion relation for the distribution in Sec. III. In Sec. IV, we calculate the growth-site probability distribution from the recursion relation, while a brief summary is given in Sec. V. Finally, we discuss different variants of the model in the Appendix.

II. MODEL

The model is defined as follows. The first generation [Fig. 1(a)], which consists of three wedges, is the generator of the model. In order to get the next generation, we replace every wedge in the first generation with the generator [Fig. 1(b)]. The third generation is obtained by replacing every wedge in the second generation with the generator [Fig. 1(c)]. In general, one can obtain generation n by replacing all the wedges generation $n-1$ with the generator.

In this paper, we consider a family of generators [10] defined by two indices (b,l). Here b is the linear size, and l the number of "empty layers" of the generator [Fig. 2(a)]. For example, the generator of Fig. 1(a) is labeled as (2,0). Since l cannot be larger than $b-2$, for each value of b there are $b-1$ models with $l=0,1,2,\ldots,b-2$. In Fig. 2(b), we show all the generators with $b=2$–4. However, these two indices do not uniquely determine the structure. For example, consider two generators in Fig. 2(c). These two structures are distinct, but have the same indices (4,0). The structure does not only depend on (b,l), but also on the way the wedges are connected. However, we will later show that *the growth-site probability distribution is uniquely determined by the two indices*, which implies that the two structures in Fig. 2(c) have the same growth-site probability distribution.

We now define the growth-site probability distribution. One launches a particle from a circle whose radius is much larger than the linear size of the model. The particle performs a random walk until it steps on one of the perimeter sites of the cluster. The growth probability p_i at perimeter site i is defined as the probability that the random walker steps on site i for the first time. The growth-site probability distribution [4] is the histogram of p_i for a given cluster. In order to simplify the calculation, we assume that the growth probability along the top line of the cluster p_0 is constant. Here, we set $p_0=1$ [11]. We also assign one growth site for the smallest wedge.

We consider three variants of the model, differing on how to approximate growth sites at "larger wedges." By "larger wedges," we mean the wedges whose linear size is larger than unity. The wedge of linear size 2 at the center in Fig. 1(b) is an example of a larger wedge. In variant A, the growth sites at larger wedges are ignored. In variant B, we assign one growth site for a wedge, regardless of the size of the wedge. In variant C, the number of growth sites is proportional to the linear size of the wedge. Since we find *no difference* for the scaling form of the growth-site probability distribution among the three variants, we present in detail only variant A in the body of this paper. The other variants are discussed in the Appendix.

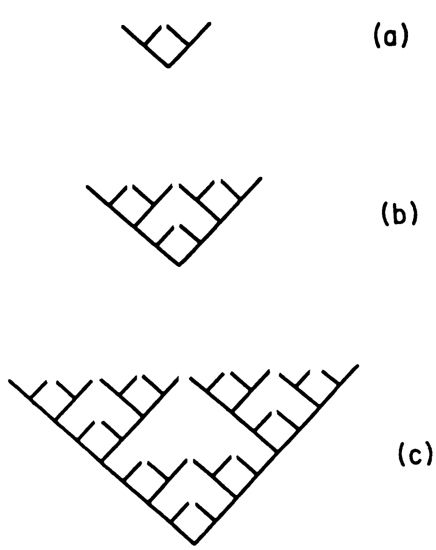

FIG. 1. Construction of the (2,0) model: (a) the generator and the first generation of the model, (b) the second generation, and (c) the third generation.

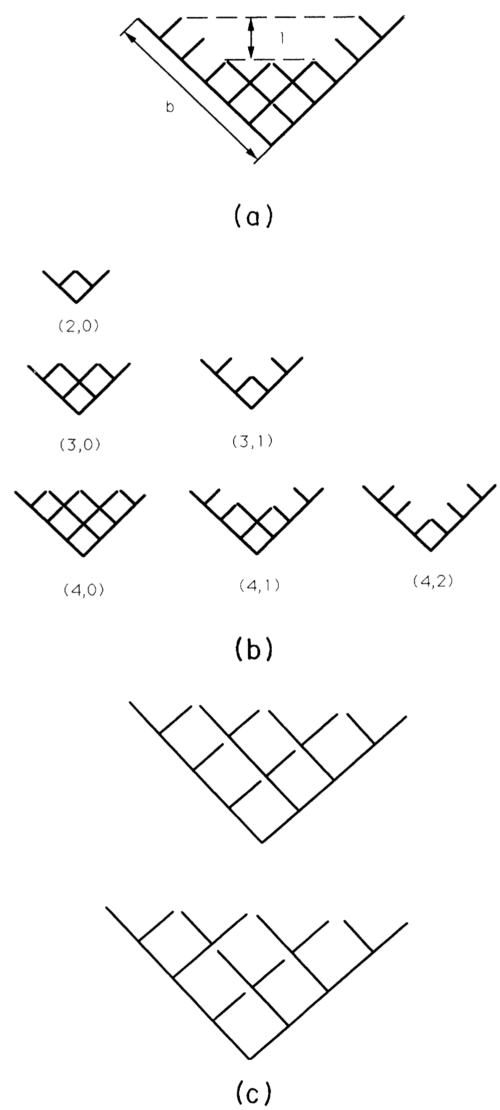

FIG. 2. The generator of the (b,l) model. (a) The generator is defined with the linear size b and the number of empty layers l. (b) All the possible generators with $b=2$–4. (c) Two generators that have the same index (4,0) but different structures.

III. RECURSION RELATIONS

In this section, we derive a recursion relation for the growth-site probability distribution. Consider the (2,0) model again (Fig. 1). The growth-site probability distribution in generation n, $D_n^{(2,0)}(x)$, is defined as

$$D_n^{(2,0)}(x)dx \equiv \mathcal{N} ,$$

where $\mathcal{N}$ represents the number of perimeter sites with $x \le x_i \le x+dx$ and $x_i \equiv -\ln p_i$.

For the first generation [Fig. 1(a)], there are three wedges. The p_i at the outer two wedges is p_0 by definition. The p_i at the inner wedge is simply a product of p_0 and the transition probability from the outer wedge to the inner wedge. The transition probability scales as $L^{-(\pi/\theta+1)}$, where L and θ are the length and angle of the wedge, respectively (see, e.g., Harris and Cohen [7], and also Ref. [9]). Since the transition probability is 1 ($L=1$), p_i at the inner wedge is also 1. Therefore, $D_1^{(2,0)}(x)=3\delta_{x,0}$.

The second generation [Fig. 1(b)] consists of three generators. The growth-site probability distribution at the outer two generators is identical to $D_1^{(2,0)}(x)$. Since the transition probability to the inner generator is $2^{-(\pi/\theta+1)}$, the distribution at the inner generator is $D_1^{(2,0)}[x-(\pi/\theta+1)\ln 2]$. Here we assume that the width of the channel is independent of n. This is not a stringent assumption, since the entire calculation is still valid if the width of the largest neck w scales as $w \sim M^{\alpha}$ provided $\alpha < 1/d_f$, where M is the mass of the aggregate [9]. After rescaling the variable $x=x\theta/(\pi+\theta)\ln 2$, we get a simple recursion relation $D_2^{(2,0)}(x)=2D_1^{(2,0)}(x)+D_1^{(2,0)}(x-1)$. The first term is due to the two generators at the outer side, and the second term is due to the generator at the inner side. In the same way, one can derive the recursion relation between $D_n^{(2,0)}(x)$ and $D_{n-1}^{(2,0)}(x)$,

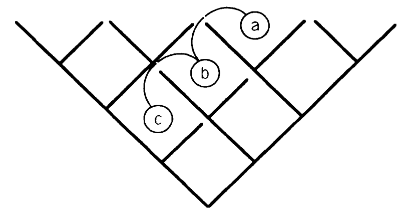

FIG. 3. The growth probability at a wedge is a function of how many wedges one must pass through to reach the top. Therefore $p_b=p_a t$, $p_c=p_a t^2$, where t is the transition probability between nearest wedges.

$$D_n^{(2,0)}(x)=2D_{n-1}^{(2,0)}(x)+D_{n-1}^{(2,0)}(x-n+1) . \tag{1}$$

We can also derive the recursion relation for the total number of growth sites $N_n^{(2,0)}$. Since generation n consists of three generations $n-1$,

$$N_n^{(2,0)}=3N_{n-1}^{(2,0)} . \tag{2}$$

We next derive a similar recursion relation for the general model (b,l). Consider wedges a,b,c inside the generator (4,0), as shown in Fig. 3. The growth probabilities at the wedges p_a, p_b, and p_c are p_0 times the products of transition probabilities between wedges. For example, $p_a=p_0$, $p_b=p_0 t$, and $p_c=p_0 t^2$, where t is the transition probability to nearest-neighbor wedges (where $t \sim L^{-(\pi/\theta+1)}$). Note that the growth probability of a wedge depends on the number of wedges it has to go through in order to reach the top of the cluster. Also, this number is independent of the way that the wedges are connected to each other. Therefore, the growth probability of a generator is uniquely determined by the indices (b,l), as claimed before.

The growth-site probability distribution for the (b,l) model can be determined from the following recursion relation:

$$\begin{aligned} D_n^{(b,l)}(x)=&2lD_{n-1}^{(b,l)}(x)+(b-l)D_{n-1}^{(b,l)}(x)+(b-l-1)D_{n-1}^{(b,l)}(x-n+1)+(b-l-2)D_{n-1}^{(b,l)}(x-2n+2)\\ &+\cdots+2D_{n-1}^{(b,l)}[x-(b-l-2)(n-1)]+D_{n-1}^{(b,0)}[x-(b-l-1)(n-1)] , \end{aligned} \tag{3}$$

with the initial condition $D_1^{(b,l)}(x)=[(b-l)(b-l+1)/2+2l]\delta_{x,0}$. Also, x is rescaled as $x\theta/(\pi+\theta)\ln b$. The recursion relation for the total number of growth sites is

$$N_n^{(b,l)}=[(b-l)(b-l+1)/2+2l]N_{n-1}^{(b,l)} . \tag{4}$$

Since the linear size of the system $L_n^{(b,l)}$ satisfies the recursion relation $L_n^{(b,l)}=bL_{n-1}^{(b,l)}$, the fractal dimension of growth site $d_g^{(b,l)}$ is

$$\begin{aligned} d_g^{(b,l)}&=\lim_{n\to\infty}\frac{\ln N_n^{(b,l)}}{\ln L_n^{(b,l)}}\\ &=\frac{\ln[(b-l)(b-l+1)/2+2l]}{\ln b} , \end{aligned} \tag{5}$$

which is the same as the fractal dimension of the cluster, and is a function of b and l.

IV. CALCULATION OF $D_n^{(b,l)}(x)$

In this section, we calculate $D_n^{(b,l)}(x)$, starting from the recursion relation (3). We first consider the (2,0) model, which has the simplest structure among the family. Then we will extend our results for the (2,0) model to the arbitrary (b,l) case.

Define the generating function $G_n^{(b,l)}$ for the distribution $D_n^{(b,l)}(z)$ as

$$G_n^{(b,l)}(z) \equiv \sum_{x=0}^{\infty} D_n^{(b,l)}(x) z^x . \tag{6}$$

When we multiply (1) by z^x, and sum over x, we obtain $G_n^{(2,0)}(z)=(2+z^{n-1})G_{n-1}^{(2,0)}(z)$. Combining this relation with the fact that $G_1^{(2,0)}(z)=3$, we obtain

$$G_n^{(2,0)}(z)=3\prod_{i=1}^{n-1}(2+z^i)\ . \tag{7}$$

The order of $G_n^{(2,0)}(z)$ is $n(n-1)/2$, and since $x\equiv-\ln p$, it follows from (6) that $\ln p_{\min}=-n(n-1)/2\sim-(\ln L)^2$.

A. Connection to partitions

Next we derive a relation between $D_n^{(2,0)}(x)$ and a certain type of partition [12]. Equation (7), combined with (6), gives $\sum_{x=0}^{\infty}D_n^{(2,0)}(x)z^x=3\times2^{n-1}\prod_{i=1}^{n-1}(1+\frac{1}{2}z^i)$. The product in this expression can be expanded to 2^{n-1} terms. Every term in the expansion is a product of $n-1$ components, and each component can be either 1 or $\frac{1}{2}z^i$. If we define an index I_i for the ith term as

$$I_i\equiv\begin{cases}0, & \text{the } i\text{th component is } 1\\ 1, & \text{the } i\text{th component is } z^i/2\ .\end{cases} \tag{8}$$

Then a term can be expressed as $2^{-\sum_{i=1}^{n-1}I_i}z^{\sum_{i=1}^{n-1}iI_i}$. Therefore, the coefficient of z^x term is $\sum_{m=0}^{n-1}2^{-m}p(\{1,2,\ldots,n-1\}\le1,m,x)$. Here $p(\{S\}\le d,m,x)$ is the number of partitions of x with m components, drawn from a set $\{S\}$, and every member of the set cannot be drawn more than d times. We now arrive at the desired relation $D_n^{(2,0)}(x)=3\times2^{n-1}\sum_{m=0}^{n-1}2^{-m}p(\{1,2,\ldots,n-1\}\le1,m,x)$. The essential reason for this connection with partitions lies in the fact that the growth probability is a function of the products of void area the random walker must pass through in order to reach a site.

Having established the connection with partitions, one may be tempted to evaluate $D_n^{(2,0)}(x)$ using residue calculus, exactly the way Hardy and Ramanujan obtained the unrestricted partition function [13]. In order to evaluate the contour integral, they used a key relation—the inversion formula. However, this relation is known only for the infinite product, which sets a limitation in this direction for calculating $D_n^{(2,0)}(x)$.

B. Gauss polynomial

We take a different approach to obtain $D_n^{(2,0)}(x)$. We first use the Cauchy identity [12] to expend $G_n^{(2,0)}(z)$,

$$G_n^{(2,0)}(z)=2^n\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}2^{-j}z^{j(j-1)/2}\ . \tag{9}$$

Here, $[{n \atop j}]$ is the Gauss polynomial, which is defined as $[{n \atop j}]\equiv(z)_n/(z)_j(z)_{n-j}$, and $(z)_n\equiv(1-z)(1-z^2)(1-z^3)\cdots(1-z^n)$. In order to calculate the distribution, one has to know the coefficient $C_n^j(k)$ of the z^k term for the Gauss polynomial $[{n \atop j}]$. Some properties of $C_n^j(k)$ are known: (i) $[{n \atop j}]$ is a $j(n-j)$th-order polynomial in z. (ii) $[{n \atop j}]$ is reciprocal, $C_n^j(k)=C_n^j(n-k)$. (iii) $[{n \atop j}]$ is unimodal. There exist m such that

$$C_n^j(0)\le C_n^j(1)\le C_n^j(2)\le\cdots\le C_n^j(m)\ge C_n^j(m+1)\ge C_n^j(m+2)\ge\cdots\ge C_n^j(jn-j^2)\ .$$

(iv) $[{n \atop j}]_{z=1}=n!/j!(n-j)!$

One can define a "normalized" Gauss polynomial $g_n^j(z)$ as

$$g_n^j(z)\equiv\begin{bmatrix}n\\ j\end{bmatrix}\Bigg/\frac{n!}{j!(n-j)!}\equiv\sum_{k=0}^{J(n-j)}c_n^j(k)z^k\ . \tag{10}$$

Here we use the work "normalized," since $g_n^j(z=1)=\sum_{k=0}^{j(n-j)}c_n^j(k)=1$. Roughly speaking, the coefficient $c_n^j(k)$ is a symmetric function of k around $j(n-j)/2$, with the maximum at the middle. Also, the total area under the curve, $\sum_k c_n^j(k)$, is unity. When we substitute (10) into (9), and use a Gaussian approximation for the binomial coefficient $2^{-j}n!/j!(n-j)!\sim\exp[-9/4n(j-n/3)^2]$, we obtain

$$G_n^{(2,0)}(z)\sim\sum_{j=0}^{n}\exp\left[-\frac{9}{4n}(j-n/3)^2\right]g_n^j(z)z^{j(j-1)/2}\ . \tag{11}$$

In order to get further information for $c_n^j(k)$, we calculate the first two moments from the normalized Gauss polynomial $g_n^j(z)$,

$$\begin{aligned}\langle1\rangle&=1\ ,\\ \langle k\rangle&=\tfrac{1}{2}j(n-j)\ ,\\ \langle k^2\rangle&=\tfrac{1}{4}j^2(n-j)^2+\tfrac{1}{12}j(n-j)(n+1)\\ &=\langle k\rangle^2+\tfrac{1}{12}j(n-j)(n+1)\ ,\end{aligned} \tag{12}$$

where $\langle A\rangle\equiv\sum_k c_n^j(k)A_k$. Since we know the average, and the width of the distribution $\langle(k-\langle k\rangle)^2\rangle=j(n-j)(n+1)/12$, we can now construct a Gaussian

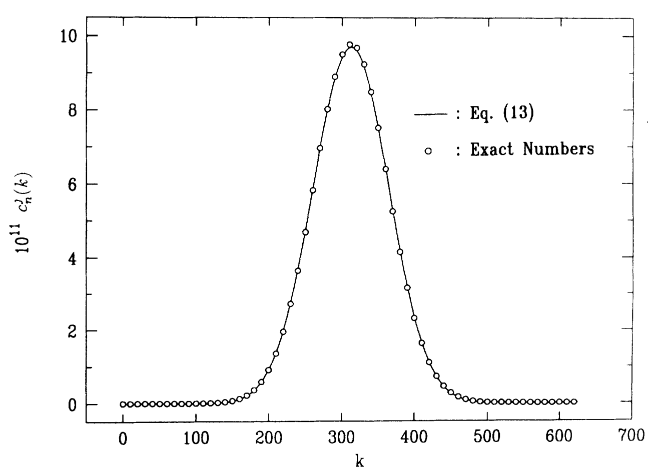

FIG. 4. Gaussian approximation for the coefficient of Gauss polynomial. The solid line is the Gauss approximation (13) for the exact numbers (marked by open circles) of $c_n^j(k)$ with $n=100$ and $j=50$.

approximation for the normalized Gauss polynomial:

$$c_n^j(k) \simeq \left[\frac{2}{j(n-j)(n+1)\pi}\right]^{1/2} \times \exp\left[-6\frac{[k-j(n-j)/2]^2}{(n+1)j(n-j)}\right] . \quad (13)$$

In order to check the validity of this approximation, we calculate $C_n^j(k)$ using the recursion relation [12], $[{n \atop m}]=[{n-1 \atop m}]+z^{n-m}[{n-1 \atop m-1}]$. In Fig. 4, we plot exact values of $C_n^j(k)$ and their Gaussian approximation [the right-hand side of (13) multiplied by $n!/j!(n-j)!$] for $n=50$ and $j=25$. We emphasize that without using any adjustable parameters, we find good agreement between the two values. Also, we find similar agreement for other values of n and j.

C. Calculation of $D_n^{(2,0)}(x)$

Using the Gaussian approximation for $c_n^j(k)$, we now proceed to evaluate the summation (11). Substituting (13) into (11), we find

$$G_n^{(2,0)}(z) \sim \sum_{j=0}^{n} \exp\left[-\frac{9}{4n}(j-n/3)^2\right] \times \sum_x \exp\left[-6\frac{[x-j(n-j)/2]^2}{(n+1)j(n-j)}\right] \times z^x z^{j(j-1)/2} , \quad (14)$$

where we dropped the constant and the power-law term in front of the summation. By redefining the summation index x to $x-j(j-1)/2$, we obtain

$$\sum_x \exp\left[-6\frac{[x-j(n-j)/2]^2}{(n+1)j(n-j)}\right] z^x z^{j(j-1)/2} = \sum_x \exp\left[-6\frac{[x'-j(n-1)/2]^2}{(n+1)j(n-j)}\right] z^x .$$

We now make one more approximation, which will be justified later. We replace $(n+1)j(n-j)$ in the denominator of the exponential with that of the dominant Gauss polynomial, $2n^3/9$ $(j=n/3)$. Since we are interested in the $n \gg 1$ behavior of the distribution, we also replace $n+1$ and $n-1$ with n. Evaluating the summation over j, we obtain

$$G_n^{(2,0)}(z) \sim \sum_x \exp\left[-\frac{27}{4n^3}\left[x-\frac{n^2}{6}\right]^2\right] z^x , \quad (15)$$

which is the key result of this paper.

In the previous derivation, we make two approximations: (i) $c_n^j(k)$ is replaced with the Gaussian with width $[A(n,j)n^3]^{1/2}$, and (ii) the amplitude $A(n,j)$ is set to be a constant. We test how sensitive the result (15) is to these approximations. If we use $\exp[-Bn^{-\alpha}(x-jn/2)^2]$ as an approximation for $c_n^j(k)$, the corresponding equations for (15) are

$$G_n^{(2,0)}(z) \sim \begin{cases} \sum_x \exp[-9n^{-3}(x-n^2/6)^2]z^x , & \alpha<3 \\ \sum_x \exp\{-[9B/(9+B)]n^{-3}(x-n^2/6)^2\}z^x , & \alpha=3 \\ \sum_x \exp[-Bn^{-\alpha}(x-n^2/6)^2]z^x , & \alpha>3 . \end{cases} \quad (16)$$

In general, the scaling form of the generating function depends *only* on the exponent α, not on the amplitude B. For $\alpha \le 3$, the scaling is *independent* of both α and B, since the Gaussian is strongly localized, and can be considered to be a δ function. Since the maximum value of α for the Gauss polynomial is 3 (the maximum value is $n^3/4$), the above approximations will change, at most, the amplitudes of the x and x^2 terms.

As an independent test, we calculate directly the moments of the distribution from its generating function. Consider a generating function $G'(z)$ [and its distribution $D'(x)$] defined as

$$G'(z) \equiv \sum_x D'(x)z^x \equiv \prod_{i=1}^{n-1} (1+z^i) , \quad (17)$$

which has exactly the same structure as $G_n^{(2,0)}(z)$, as shown in (7). In fact, the asymptotic form for $D'(x)$, obtained by following the same analysis, is identical to (15) except for the amplitudes of x^2 and x. The first four moments of $D'(x)$ are

$$\begin{aligned} \langle 1\rangle &= 1+ , \\ \langle x\rangle &= \tfrac{1}{4}n(n-1) , \\ \langle x^2\rangle &= \tfrac{1}{16}n^2(n-1)^2+\tfrac{1}{24}n(n-1)(2n-1)=\langle x\rangle^2+\tfrac{1}{24}n(n-1)(2n-1) , \\ \langle x^3\rangle &= \tfrac{1}{64}n^2(n-1)^2(n^2+3n-2)=\langle x\rangle^3+3\langle x\rangle\langle \delta^2 x\rangle , \\ \langle x^4\rangle &= [\tfrac{1}{4}n(n-1)]^4+6[\tfrac{1}{4}n(n-1)]^2\frac{1}{24}n(n-1)(2n-1)+3[\tfrac{1}{24}n(n-1)(2n-1)]^2-\tfrac{1}{240}n(n-1)(2n-1)(3n^2-3n-1) \\ &= \langle x\rangle^4+6\langle \delta^2 x\rangle\langle x\rangle^2+3\langle \delta^2 x\rangle^2[1-O(1/n)] , \end{aligned} \quad (18)$$

where $\langle A \rangle \equiv \sum_x D'(x) A(x) / \sum_x D'(x)$, and $\delta x \equiv x - \langle x \rangle$. Note that these moments are the same as those for the Gaussian distribution,

$$D'(x) \sim \exp\left[-12\frac{[x-\frac{1}{4}n(n-1)]^2}{n(n-1)(2n+1)}\right] \simeq \exp\left[-\frac{6}{n^3}(x-\tfrac{1}{4}n^2)^2\right], \tag{19}$$

which is the same form as (15) except for the amplitudes, as discussed before. Thereby we establish the scaling form for the distribution

$$D_n^{(2,0)}(x) \sim \exp\left[-\frac{3}{2A^{(2,0)}n^3}\left[x-\frac{B^{(2,0)}}{2}n^2\right]^2\right], \tag{20}$$

where $A^{(2,0)}$ and $B^{(2,0)}$ are amplitudes.

D. Calculation of $D_n^{(b,l)}(x)$

We now calculate the scaling form of the distribution for the (b,l) model, based on the result (20) obtained for the (2,0) model. The generating function $G_n^{(b,l)}(z)$ can be calculated by combining the definition (6) and the recursion relation (3),

$$G_n^{(b,l)}(z) = G_1^{(b,l)}(z) \prod_{j=1}^{n-1} [2l + (b-l) + (b-l-1)z^j + (b-l-2)z^{2j} + \cdots + z^{(b-l-1)j}], \tag{21}$$

where $G_1^{(b,l)}(z) = [(b-l)(b-l+1)/2 + 2l]$. In (21), the term inside the square bracket is a $(b-l-1)$th-order polynomial of z^j, and can be factorized into $b-l-1$ polynomials of order z^j. From (21), the order of the polynomial is $(b-l-1)n(n-1)/2$, so $\ln p_{\min} \sim -(\ln L)^2$.

Equation (21) can be written as

$$G_n^{(b,l)}(z) \equiv (b+l)^{n-1} \prod_{j=1}^{n-1} \prod_{m=1}^{b-l-1} (1+t_m z^j), \tag{22}$$

where t_m are complex numbers dependent upon b and l. Using the Cauchy identity (9) and the Gaussian approximation for the Gauss polynomial (13), Eq. (22) becomes

$$G_n^{(b,l)}(z) \sim \prod_{m=1}^{b-l-1} \sum_{j_m} \sum_{k_m} \exp\left[-\frac{(1+t_m)^2}{2t_m n}\left[j_m - \frac{n t_m}{1+t_m}\right]^2 - \frac{6(1+t_m)^2}{t_m n^3} \times \left[k_m - \frac{n j_m}{2}\right]^2\right] z^{k_m}. \tag{23}$$

After evaluating the summation over j_m, we obtain

$$G_n^{(b,l)}(z) \sim \prod_{m=1}^{b-l-1} \sum_{k_m} \exp\left[-\frac{3(1+t_m)^2}{2nt_m} \times \left[k_m - \frac{t_m}{2(1+t_m)}n^2\right]^2\right] \times z^{k_m}, \tag{24}$$

where we dropped the constant and the power-law terms in front of the summation. The products over m can be rewritten as

$$G_n^{(b,l)}(z) \sim \sum_x \sum_{k_1} \cdots \sum_{k_{b-l-1}}{}' \exp\left[-\sum_m \frac{3(1+t_m)^2}{2t_m n^3}\left[k_m - \frac{t_m}{2(1+t_m)}n^2\right]^2\right] z^x, \tag{25}$$

where $\sum'_{k_1,k_2,\ldots,k_{b-l-1}}$ means $\sum_{k_1}\sum_{k_2}\cdots\sum_{k_{b-l-1}}$ with the restriction $\sum_m k_m = x$. After we evaluate the multiple summation over k_m,

$$G_n^{(b,l)}(z) \sim \sum_x \exp\left[-\frac{3}{2A^{(b,l)}n^3}\left[x - \frac{n^2}{2}B^{(b,l)}\right]^2\right] z^x. \tag{26}$$

Here $A^{(b,l)} \equiv \sum_{m=1}^{b-l-1} t_m/(1+t_m)^2$, and

$$B^{(b,l)} \equiv \sum_{m=1}^{b-l-1} t_m/(1+t_m) = (b-l-1) - 4(b-1)(b-l-1)/[2(b+l)+(b-l)(b-l-1)].$$

Therefore, the scaling form obtained (20) holds not only for the (2,0) model but also for the entire family of models (b,l) considered here. We emphasize that even if the form (26) is correct, the amplitudes $A^{(b,l)}$ and $B^{(b,l)}$ determined above are only approximate values.

Using the recursion relation (3), we also calculate the numerical values of the distribution $D_n^{(b,l)}(x)$. In Fig. 5, we plot both the numerical values and the analytic expression (26) for (2,0) and (3,0) models with $n=100$. The amplitudes, determined from the expressions given above, are $A^{30}(2,0)=\frac{2}{9}$, $b^{(2,0)}=\frac{1}{3}$ for the (2,0) model, and $A^{(3,0)}=\frac{5}{9}$, $B^{(3,0)}=\frac{2}{3}$ for the (3,0) model, respectively. Also, the maximum values are normalized to unity. One can see a good agreement between the two, even for the amplitudes— the position and width of the peak. We also find good agreement for other values of b and l.

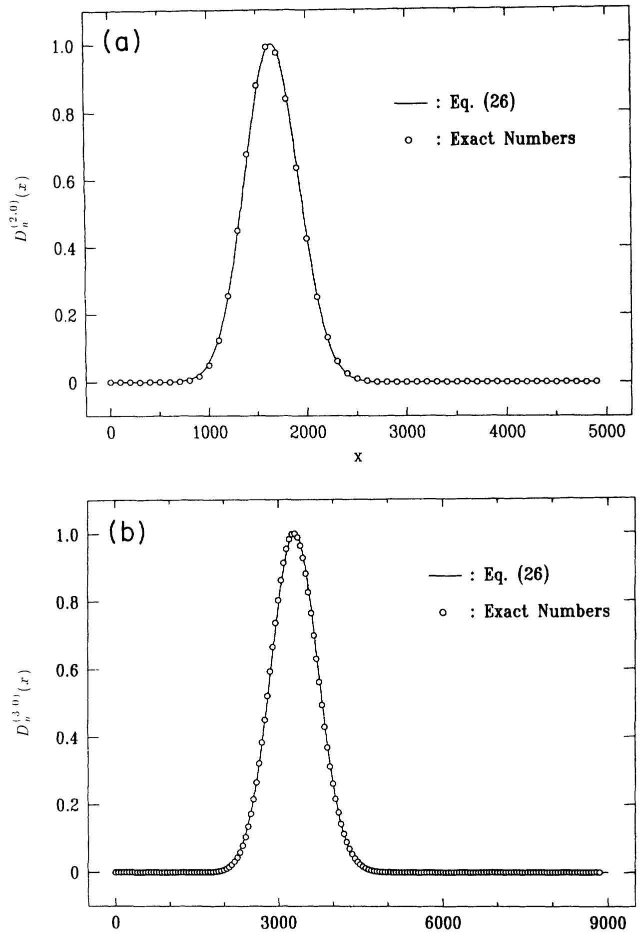

FIG. 5. Comparison of the scaling form (26), shown in solid lines, with the exact numbers (○) with $n=100$: (a) the (2,0) model, and (b) the (3,0) model.

V. DISCUSSION AND SUMMARY

We have proposed a family of hierarchical models for the structure of DLA. The key feature in the model is a hierarchy of self-similar voids connected by narrow channels. In order to obtain the growth-site probability distribution $n(\alpha,M)$, we first expand the distribution, using the Cauchy identity, in terms of Gauss polynomials. Each distribution for one Gauss polynomial "marginally" overlaps with all the other terms. This allows us to resume the expansion to get a closed form for $n(\alpha,M)$. Due to the "marginal" overlap, we cannot get the exact amplitudes for the distribution. However, we find approximate values, which are in good agreement with exact numerical data.

One interesting point to emerge is that the distribution is the same (excluding amplitudes) for the entire family of models studied here. Since the common ingredient for the entire family is a hierarchy of self-similar "voids" separated by narrow channels, it is possible that the form of $n(\alpha,M)$ obtained here is just a consequence of the void-channel feature, and is independent of further details of a model.

We now discuss a possible connection with "real" DLA clusters. Schwarzer *et al.* [8(b)] calculated $n(\alpha,M)$ for off-lattice DLA clusters of $M\le 20\,000$, and found a scaling form $n(\alpha,M)\sim\exp[-A(\ln M)^{-\delta}\alpha^{\gamma}]$ with $\gamma=2.0\pm0.3$ and $\delta=1.3\pm0.3$. The family of models studied here has the same form with $\gamma=2$ and $\delta=1$, which is within the confidence limits of the numerical results of Ref. [8(b)]. This fact, together with the corresponding agreement for $p_{\min}$ [8], gives additional support for the "void-channel" distribution of DLA clusters. The "void-channel" concept should be further tested by studying directly the geometry of "real" DLA clusters (see, e.g., Ref. [7]), and should be derived from the microscopic growth rules.

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APPENDIX: OTHER VARIANTS OF THE MODEL

As discussed in Sec. II, we have developed three distinct variants (A, B, and C) of the model. In Ref. [9], different variants are studied using numerical methods, and no essential differences are found for the growth-site probability distribution. Therefore, in this paper, we concentrated mostly on the most simple variant—variant A. In this Appendix, we will present some analytic results for the growth-site probability distribution and the number of growth sites for other variants of the model.

We start with variant B. Since we assign one growth site for every wedge, the recursion relation of the distribution $\tilde D_n^{(b,l)}(x)$, which corresponds to (3), becomes

$$\begin{aligned}\tilde D_n^{(b,l)}(x)=&\,2l\tilde D_{n-1}^{(b,l)}(x)+(b-l)\tilde D_{n-1}^{(b,l)}(x)+(b-l-1)\tilde D_{n-1}^{(b,l)}(x-n+1)+(b-l-2)\tilde D_{n-1}^{(b,l)}(x-2n+2)\\&+\cdots+2\tilde D_{n-1}^{(b,l)}[x-(b-l-2)(n-1)]+\tilde D_{n-1}^{(b,0)}[x-(b-l-1)(n-1)]\\&+(b-l-1)\delta_{x,n-1}+(b-l-2)\delta_{x,2(n-1)}+\cdots+2\delta_{x,(b-l-2)(n-1)}+\delta_{x,(b-l-1)(n-1)}\ .\end{aligned}\qquad \text{(A1)}$$

The δ functions are contributions from the growth sites in larger wedges. Also, the initial condition is modified to $\tilde D_1^{(b,l)}(x)=[(b-l)^2+2l]\delta_{x,0}$. We define the generating function $\tilde G_n^{(b,l)}(x)\equiv\sum_x\tilde D_n^{(b,l)}(x)z^x$. If we substitute this definition to (A1), we obtain

$$\tilde G_n^{(b,l)}(z)=\tilde G_{n-1}^{(b,l)}(z)f_{n-1}^{(b,l)}(z)+f_{n-1}^{(b,l)}(z)-b-l\ . \qquad \text{(A2)}$$

Here $\tilde G_1^{(b,l)}(z)=(b-l)^2+2l$, and $f_n^{(b,l)}(z)\equiv(b+l)+(b-l-1)z^n+(b-l-2)z^{2n}+\cdots+z^{(b-l-1)n}$. Substituting a "trial" solution $\tilde{G}_n^{(b,l)}(z) \equiv h_n^{(b,l)}(z)\tilde{G}_1^{(b,l)}(z)\prod_{i=1}^{n-1} f_i^{(b,l)}(z)$ to (A2), we obtain a recursion relation for function $h_n^{(b,l)}(z)$

$$h_n^{(b,l)}(z) = h_{n-1}^{(b,l)}(z) + \frac{f_{n-1}^{(b,l)}(z) - b - l}{\tilde{G}_1(z)\prod_{i=1}^{n-1} f_i^{(b,l)}(z)} , \tag{A3}$$

where $f_1^{(b,l)}(z) = 1$. The solution of (A3) is

$$h_n^{(b,l)}(z) = 1 + \sum_{i=1}^{n-1} [f_i^{(b,l)}(z) - b - l] \Bigg/ \left[\tilde{G}_1^{(b,l)}(z)\prod_{j=1}^{n-1} f_j^{(b,l)}(z)\right] . \tag{A4}$$

Combining this with the definition of $h_n^{(b,l)}(z)$, we arrive at

$$\tilde{G}_n^{(b,l)}(z) = \tilde{G}_1^{(b,l)} \prod_{i=1}^{n-1} f_i^{(b,l)}(z) \left[1 + \sum_{j=1}^{n-1} \frac{f_j^{(b,l)}(z) - b - l}{\tilde{G}_1^{(b,l)} \prod_{k=1}^{j} f_k^{(b,l)}(z)}\right] . \tag{A5}$$

We now study the structure of $\tilde{G}_n^{(b,l)}(z)$. The product term outside the bracket in (A5) is exactly the generating function of variant A. Inside the bracket, after the leading term 1, there are series of terms $[f_j^{(b,l)}(z) - b - l]/\prod_{k=1}^{j} f_k^{(b,l)}(z)$. The total contribution of each term, calculated as its value at $z=1$, is $[(b-l)(b-l-1)/2+b+l]/[(b-l)(b-l-1)/2]^k$, which is an exponentially decaying function of k. We can also calculate the coefficient of the polynomial $1/\prod_{k=1}^{j} f_k^{(b,l)}(z)$. First, we use another form of Cauchy identity [12],

$$\prod_{i=0}^{n-1} \frac{1}{(1+tz^i)} = \sum_{j=0}^{\infty} \begin{bmatrix} n+j-2 \\ j \end{bmatrix} (-t)^j$$
$$= \sum_{j=0}^{\infty} \frac{(n+j-2)!}{(n-2)!j!} g_{n+j-2}^{j}(z)(-t)^j , \tag{A6}$$

where $g_{n+j-2}^{j}(z)$ is the normalized Gauss polynomial defined in Sec. IV B. We now use two approximations, as explained in Sec. IV B, $(n+j-2)!/(n-2)!j! \sim \exp\{-[j-n(1+\ln 2t)]^2/2n\}$, and $g_{n+j-2}^{j}(z) \sim \sum_x \exp\{-6(x-nj/2)^2/[jn(n+j)]\}z^x$. Substituting these approximations into (A6),

$$\prod_{i=1}^{n-1} \frac{1}{1+tz^i} \sim \sum_{j=0}^{\infty}\sum_x (-1)^j \exp \left[-\frac{1}{2n}[j-n(1+\ln 2t)]^2 - 6\frac{(x-nj/2)^2}{jn(n+j)}\right] z^x . \tag{A7}$$

Equation (A7) is similar to (16) except for the phase factor $(-1)^j$. However, due to this factor, the summand has an alternating sign, which causes the entire summation to almost cancel out (a kind of "destructive interference"), resulting in the exponentially decaying contribution mentioned above. The coefficient of $\prod_{k=1}^{j}[f_k^{(b,l)}(z)]^{-1}$ can be calculated using

$$f_n^{(b,l)}(z) = \prod_{i=1}^{b-l-1} (1+t_i z^n) , \tag{A8}$$

as shown in Sec. IV D.

Having obtained the coefficients for the polynomial $\prod_{k=1}^{j}[f_k^{(b,l)}(z)]^{-1}$, we now return to discuss (A5). As noted before, the summand inside the large parentheses is an exponentially decaying function of the summation variable k. Therefore, the entire summation can be replaced with a finite number of terms. Compared to variant A, the generating function of variant B has additional terms (the summation inside the large parentheses), and since these terms are finite, it cannot change the asymptotic behavior $(n\rightarrow\infty)$ of the generating function. The same argument can be applied to variant C, since the only difference is in the coefficients of the summand inside the large parentheses. Therefore, the scaling form of the distribution function (26) is the same for other variants (B and C) of the model.

We conclude the Appendix by calculating the fractal dimension for the number of growth sites. The recursion relation for the number of growth sites $\tilde{N}_n^{(b,l)}$ for variant B is

$$\tilde{N}_n^{(b,l)} = [\tfrac{1}{2}(b-l)(b-l-1)+2l]\tilde{N}_{n-1}^{(b,l)} + \tfrac{1}{2}(b-l-1)(b-l) . \tag{A9}$$

The number of growth sites $\tilde{N}_n^{(b,l)}$, which can be obtained by using the same method to solve (A2), is

$$\tilde{N}_n^{(b,l)} = \frac{1}{A-1}\{A^{n-1}[(A-1)N_1^{(b,l)}+B]-B\} , \tag{A10}$$

where $A \equiv (b-l)(b-l-1)/2+2l$ and $B \equiv (b-l)(b-l-1)/2$, and $N_1^{(b,l)} = (b-l)^2+2l$. The number of growth sites $\hat{N}_n^{(b,l)}$ for variant C can also be calculated in the same way:

$$\hat{N}_n^{(b,l)} = \frac{B}{N_1^{(b,l)}(A-b)}\tilde{N}_n^{(b,l)} . \tag{A11}$$

Therefore, the other variants of the model have the same fractal dimension for the number of growth sites given by (5), which again is the same as the fractal dimension of the cluster.

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