Physica A 427 (2015) 302–325

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

The cascading vulnerability of the directed and weighted network

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HIGHLIGHTS

- The cascading failures based on directed and weighted network is studied.
- The 'load-capacity' model of the directed and weighted network is built.
- The 'over-loading' and the 'short-loading' cascading failure models based on the directed and weighted network were built.
- The power exponent β of 'load-capacity' function should be taken value (0, 1) for a good robustness of the power law network.
- The power exponent β of loading function should be taken value (0, 3) for a good robustness of the Poisson network.

ARTICLE INFO

Article history:
Received 1 June 2014
Received in revised form 27 November 2014
Available online 10 February 2015

Keywords:
Directed and weighted network
Cascading failure
Model
Vulnerability

ABSTRACT

The cascading failure can bring a huge loss for most real-world networks; but, we cannot uncover fully the mechanism and law of the cascading events occurrence. Most networks in which the cascading failure occurred are based on the various 'flows', such as power, oils, and information; moreover, the same link degree of the different nodes likely contain the different meanings, where some are large pivotal nodes and some are mini switching centers. Thus, these networks must be described by the directed and weighted network model. Besides, the 'over-loading' cascading failures were more analyzed and studied; but the cascading failures caused by 'short-loading' were less studied relatively. However, for some directed networks, such as power grids, oil pipe nets, gas pipe nets and information networks, the large-scale failures of network nodes substantially could be induced by 'short-loading' in a such similar way as 'over-loading'. Based on the above reasons, in this paper, we first built the 'load-capacity' model of the directed and weighted network. Afterwards the 'over-loading' cascading failure model and the 'short-loading' cascading failure model based on the directed and weighted network were built. Meanwhile, applying the models to two typical real networks – Poisson distribution network and power law distribution network – intensive study and numerical analysis were carried out. Lastly, two classical networks simulation experiment results are provided. After the numerical and simulation analyses, we gained the following conclusions. For the power law network, the power exponent β of 'load-capacity' function should be taken value (0, 1) for a good robustness, and the minimum in-degree and out-degree should be increased respectively, meanwhile, the weight and the scaling exponents of the in-degree and the out-degree distributions

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http://dx.doi.org/10.1016/j.physa.2015.02.035
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should be increased synchronously in the interval (2, 3) for enhancing the resistibility of 'over-loading' and 'short-loading' failures. For the Poisson network, the power exponent β of loading function should be taken value (0, 3) for a good robustness, and the average weight and the average in-degree should be increased respectively restricting $2 < \beta < 3$ for enhancing the resistibility of 'over-loading' and 'short-loading' failures.
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1. Introduction

Dr. R. Albert et al., publishing in a 2000 NATURE article, 'Error and attack tolerance of complex networks' [1], pioneered the research for the vulnerability of complex networks. They found that, the scale-free network has high robustness for the accidental failure, but behaves extremely vulnerable for the deliberate attack. On the contrary, the vulnerability of exponential network has no significant difference for the accidental failure and deliberate attack. In the same year, D.S. Callaway et al. wrote in a physics journal PRL article, 'Network robustness and fragility: Percolation on random graphs' [2]. Using the generating function method presented in literature [3], they found the critical condition of the removed node proportion when the network with arbitrary degree distribution function was paralyzed for failures or deliberate attacks. In 2001, R. Cohen et al. wrote in a PRL article, 'Breakdown of the Internet under intentional attack' [4]. With the both analysis and simulation methods to verify each other, they proposed the quantitative relationship among the scaling exponent α ($\alpha \geqslant 2$), the scale of the maximum interconnection-group and the critical proportion of the removed nodes when the scale-free network ($p(k) \sim k^{-\alpha}$) was paralyzed as a result of deliberate attacks. They concluded that the bigger the scaling exponent was, the more vulnerable the network was in deliberate attack mode. Since 2002, some scholars such as A.E. Motter [5,6], Y. Lai [5,7], I. Dobson [8,9], P. Crucitti [10,11], H.J. Sun [12], and L.D. Dueñas-Osorio [13], have paid more attention to the cascading failure phenomenon of network and set to an in-depth study. They think that the cascading failure is the main reason of a large-scale break down of the real-world networks such as power grids and Internet. They found the cascading failure that is the individual node failure caused by disaster or contingency leads to the loading diversion from this failure node to others; the loading of the nodes accepted extra loading rise suddenly, which can cause over loading of some nodes then failures; this process progressed continually will finally lead to a large-scale break down of the network. Explaining and simulating this phenomenon, they proposed the loading and capacity models based on node betweenness [5–7,12], the redistribution model CASCADE based on stationary loading P [8,9], the time-varying model which represents the efficiency decline in proportion to over loading instead of removing the over loading nodes, and calculates loading and capacity by node betweenness [10,11], and the statistic model based on historical real-world data [13]. Above papers respectively concluded that the node attack based on maximum load should be the most easy way to cause the scale-free network to cascading paralysis [5,8,6,7,10,11,9] ; removing the minimum loading node can reduce the scale of paralysis to a certain extent [6]; the load capacity coefficient will greatly affect the efficiency decreasing speed of the cascading nodes [10]; and to enhance the robustness of the network, network engineers cannot just rely on increasing the capacities of nodes and load-edges, but need adjust the network topology structure [13]. However, all of the above-mentioned models need betweenness to calculate the node loading and capacity, and calculating betweenness must obtain the global topological information of the network, which is very difficult for large-scale networks (for instance many real-world networks). Therefore, since 2008, many scholars have attempted to explore the way to calculate the node loading and capacity by the local information of nodes, and build the loading and capacity calculation models based on the local information of nodes [14–16,13,17–19]. The above main calculation models include: the loading and capacity model based on the node degree power function [14,18], the edge-weight loading and capacity model based on the degree product power function of the two ends nodes [15], the loading and capacity model based on time-varying edge-weight power function [16], the loading and capacity model based on the product power of node degree and the degree sum of its secondary neighbors [17] or the degree sum power product of multilevel neighbors [19], etc. In this period, some scholars even use the physical models such as Ohm's law and Kirchhoff conservation law to research the cascading vulnerability of the large-scale network based on 'flow' like electric power [20–22]. They gained counter-intuition conclusion that the incremental increase of one edge's load capacity could expand the cascading vulnerability of the network.

In 2010, S.V. Buldyrev et al. published an article 'Catastrophic cascade of failures in interdependent networks' in NATURE. They analyzed for the first time the cascading vulnerability of the interconnected and interdependent networks by complex network method. They found the vulnerability characteristic of interdependent networks different from the single network. This characteristic is that for the accidental failure or deliberate attack, the wider the degree distribution of interdependent networks is, the more vulnerable the interdependent networks become. But for single network, the wider the degree is, the more robust the network becomes [23]. In 2012, Dr. J. Shao et al. wrote in a NATURE PHYSICS article 'Networks formed from interdependent networks'. In this paper, the vulnerabilities of two partially interdependent networks and more interdependent NoN (Network of Networks) were analyzed systematically. They also proposed the explicit function between the removed nodes proportion and the maximum interconnection-group scale of three typical NoN with similar star, tree or chain structure, which were composed of Erdös–Rényi network [24]. In 2013, Dr. X. Huang et al. designed a bank-network bipartite graph model based on bank and bank assets, and built a cascading failure model of the financial risk transmission to predict and control the financial risks [25]. In the same year, Dr. A. Bashan et al. studied the vulnerabilities of the interdependent networks embedded spatial properties, which was more consistent with reality. They found that there was not a critical threshold of the interdependent intensity for the vulnerability transmission of the interdependent networks embedded spatial properties, and any small range failure would bring disastrous consequences [26]. In 2014, Dr. G. Dong et al. studied the vulnerabilities of the clustered networks with partial support and dependency relation in different attack strategies. They found that with the increases of the average degree or clustering coefficient, the first-order phase transition region became smaller, while the second-order phase transition region became larger; the clustering coefficient had a significant impact on the robustness of the strong coupling system, while the weak coupling system was less affected [27].

Thus, since 2000, the complex network vulnerability study is always a frontier research for the experts and scholars of physics and complex network field [1–5,8,6,7,10,11,9,14,15,12,16,13,17,20,21,23,24,28,18,19,22,29,25–27]. From the vulnerability analysis of single network to interdependent NoN vulnerability studies, from the single random failure and deliberate attack to the various attack strategies analysis, from the static vulnerability analysis of the network to the dynamic cascading failure research and so on, the studies in this direction have made significant progress and obtained a very fruitful research results.

However, for the real-world networks with cascading failures occurred, such as power grids, and Internet, the importance, status and role of the nodes with the same connectivity are usually significant difference. Some are large-scale high-voltage power distribution center or large-scale information exchange centers, while others are normal distribution stations or ordinary switches. So it cannot represent the nodes reasonably only based on a single connectivity. In addition, for the cascading failure propagation caused by the flow redistribution due to over-loading, the energy flow (such as power grids), information flow (such as Internet) and material flow (such as logistics network) carried by the network are all directed. The model describing such network should deservedly be directed network.

Compared to the vulnerability analysis based on the undirected and unweighted network widely popular now, especially the cascading failure analysis based on flow over-loading, the studies about cascading failure vulnerability based on directed and weighted network will have more strongly practical demands. Based on this, we proposed the research project for cascading failure vulnerability analysis based on directed and weighted network and carried out a relatively systematic study. At the same time, we got some valuable findings and conclusions.

This paper consists of five parts. This section is the first part, in which the current research progress of complex network vulnerability was reviewed and analyzed, and the research background was stated. The second part is 'problem statement', in which some research questions about cascading failure vulnerability analysis of directed and weighted network are defined, and the physical mechanism of cascading failures is analyzed briefly. The third part is 'the cascading failure model of the directed and weighted network', in which the cascading failure mechanism of the directed network is analyzed and studied systematically, moreover, the 'over-loading' and the 'short-loading' cascading failure models of the directed and weighted network are built. The fourth part is 'cascading failure vulnerability analysis of two typical directed and weighted networks', in which the cascading failure vulnerabilities of two typical real networks – power law network and Poisson network – are analyzed. The last part of this paper is 'conclusions and recommendations', which gives the main research results, conclusions and recommendations.

2. Problem statement

The cascading failures based on 'flow' occur in real-world network, mostly because the malfunction of individual node or edge of the network for various 'accidents' causes the 'flow' redistribution of this node or edge, which led to some redistribution nodes or edges failures for 'over-loading', and simultaneously result in another part of nodes or edges that should receive these 'flow' successive failures for 'short-loading' ('supply' shortage). This chain of failure process incessantly occurs, lastly resulting in this network 'paralyzed'. Typically, such a kind of network catastrophic failure leading to serious consequences is called a chain or cascading failures. Thus, the essential reason of such catastrophic cascading failure based on 'flow' is the circulating recurrence of the process that the 'flow' from 'upstream' nodes of the failure nodes/edges would be diverted and the 'flow' into 'downstream' nodes of the failure nodes/edges would be intermitted happen at the same time after accidental node/edge failures of the network took place. However, it is worthy of notice that the flow redistribution of an accidental failure node does not happen in the all adjacent nodes as represented in the undirected network models shown in Fig. 1, but occurs in the upstream nodes of this failure node, i.e. the adjacent nodes that are named as the 'in-degree' nodes of the directed network, shown in Fig. 2. Moreover, the direct downstream adjacent nodes of this failure node, i.e. the adjacent nodes that are named as the 'out-degree' nodes of the directed network, will not receive the 'flow' from this failure node again, shown in Fig. 2.

Furthermore, the cascading failure of most networks (such as electric power networks, oil/gas pipe networks, and telecom networks) occurs since the flow redistribution of the failure nodes results in the more 'over-loading' failures of the associated nodes, which further induces large numbers of downstream nodes disability/inefficiency for the 'intermitted flow' ('short-loading'). However, the failure nodes for short-loading are neglected in the most of current cascading failure models, and they are not counted cascading failure nodes.

This shows that, the undirected and unweighted network may not be an ideal network model for the research and analysis of a great deal of real-world cascading failures. Moreover, unlike the flow redistribution from upstream nodes of the

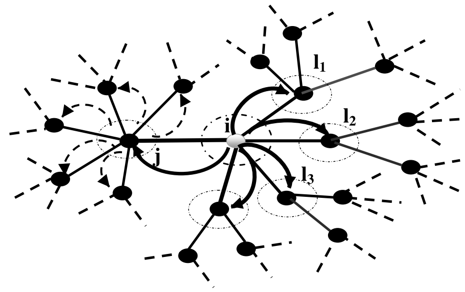

Fig. 1. The flow redistribution after an undirected network node failure.

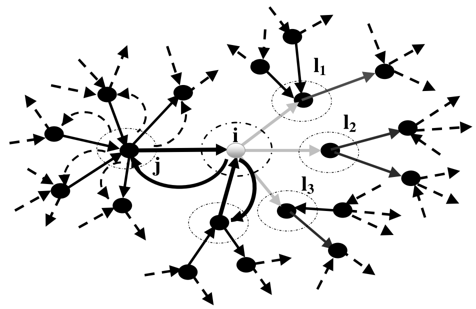

Fig. 2. The flow redistribution after a directed network node failure.

over-loading failure nodes, the short-loading failure occur in the downstream nodes of the over-loading failure nodes, which is marked chiefly by 'intermitted flow' and 'supply shortage'. It results in the nodes disability for supply shortage. Such disabled nodes sometimes constitute the principal component of the cascading failure nodes in numerous actual networks. Unfortunately, they are often neglected in current theory models and academic researches.

The two classes of the large-scale 'over-loading' and 'short-loading' cascaded failures problems of the network based on 'flow' caused by failed node that lastly result in this network 'paralyzed' constitute the main research problem of this paper. At the same time, how to model, research and analyze this two classes of problems constitutes the content of this paper. Exploring the root causes of the phenomenon, the process and the development law is the main purpose and objectives of this paper.

3. Cascading failure model of the directed and weighted network

3.1. Formal description of the directed and weighted network

To state the problem clearly, we use the following symbols and expressions to represent the directed and weighted network.

Hypothesizing the directed and weighted network is composed of N nodes, the nodes set is defined as $A = \{a_1, a_2, a_3, \ldots, a_N\}$. The nodes weight set is $W_A = \{w_1^a, w_2^a, w_3^a, \ldots, w_N^a\}$. The set of directed edges among the N nodes is $E = \{e_{11}, e_{12}, e_{13}, \ldots, e_{ij}, \ldots, e_{ji}, \ldots, e_{NN}\}$. These edges weight set is $W_E = \{w_{11}^e, w_{12}^e, w_{13}^e, \ldots, w_{ij}^e, \ldots, w_{ji}^e, \ldots, w_{NN}^e\}$. The nodes in-degree set is $K_A^{IN} = \{k_1^{in}, k_2^{in}, k_3^{in}, \ldots, k_N^{in}\}$, and the out-degree set is $K_A^{OUT} = \{k_1^{out}, k_2^{out}, k_3^{out}, \ldots, k_N^{out}\}$. The nodes in-degree probabilistic distribution function is $f^{in}(k)$, the out-degree probabilistic distribution function is $f^{out}(k)$, the node weight probabilistic distribution function is $f^a(w)$, the edge weight probabilistic distribution function is $f^e(w)$. Γ_i^{in} is the upstream adjacent node set of node i—the adjacent node set which is named as the in-degree node set of node i. Γ_i^{out} is the downstream adjacent node set of node i—the adjacent node set which is named as the out-degree node set of node i.

3.2. 'Loading' model and 'capacity' model of the directed and weighted network

The cascading failure mechanism of directed and weighted network based on 'flow' is as follows: Various accidents or deliberate attacks lead one node failure, thereby causing the 'being broken' of the flow (including various material, energy and information flows) passing through this node. The 'in-flow' from the upstream nodes has to change directions (bypass this node), and the 'out-flow' from this node to the downstream nodes will be intermitted. So, this causes the flow redistribution of upstream nodes, which diverts the over-loading of the failure node to its upstream nodes and their other downstream nodes. This further causes over-loading failures of the upstream nodes and their other downstream nodes for load over their capability. This process progressed continually until every flow redistribution node is not over its load capability, thereby leading a majority of the network over-loading cascading failures. At the same time, the downstream nodes of the failure nodes whose out-flows are shut off are confronted with supply deficiency. They will be disabled for the 'short-loading' of various material, energy and information flows. The above two classes of failures which take place concurrently will cause more large-scale 'over-loading' and 'short-loading' failures in the whole network. It will cause the network a large scale functional paralysis, as shown in Fig. 2.

The reasonable description and modeling of each node 'loading' and 'capacity', 'flow' redistribution mechanism of the failure node and failure mechanism of 'over-loading' and 'short-loading' are three key factors for correctly building above realistic failure process model.

In this section, the nodes 'loading' and the nodes 'capacity' will be represented and modeled. In the next two the redistribution mechanism of the failure node 'flow' and failure mechanism of 'over-loading' and 'short-loading' will be represented and modeled.

The preceding literature review of this paper reports that the present description and modeling methods for the network 'loading' and 'capacity' are mainly divided into two classes. One class is the modeling method based on the betweenness of nodes/edges that we call it 'loading' modeling and 'capacity' modeling method based on global information of network; another class is the modeling method of node degree and some of its neighbor information that we call it 'loading' modeling and 'capacity' modeling method based on local information of network. For many real-world networks, it is very difficult to get the global information of nodes for the large scale and complex structure or the high cost. In many cases, it is also not an absolutely necessary choice condition. So, we will prefer choosing second class of the 'loading' modeling and 'capacity' modeling method based on local information of network to first class of modeling method in this paper. Now this class of the modeling method mainly includes:

A. 'Loading' modeling and 'capacity' modeling method based on node degree power function.

The model is as follows [18]:

$$L_i = k_i^{\beta}, \quad i = 1, \dots, N \tag{1}$$

$$C_i = (1+\alpha)L_i, \quad i = 1, \dots, N. \tag{2}$$

In formula (1) and (2), L_i is the loading and C_i is the capacity of node i. The β is a tunable parameter where $\beta > 0$ and the α is a tolerance parameter where $\alpha > 0$. The k_i is the degree of node i.

B. 'Loading' modeling and 'capacity' modeling method based on the product power function of node degree and degree sum of its secondary neighbors.

The model is as follows [17]:

$$L_i = \left(k_i \sum_{j \in \Gamma_i} k_j\right)^{\beta}, \quad i = 1, \dots, N \tag{3}$$

$$C_i = TL_i, \quad i = 1, \dots, N. \tag{4}$$

In formula (3) and (4), L_i is the loading and C_i is the capacity of node i. The β is a tunable parameter where $\beta > 0$ and the T is a tolerance parameter where $T \geq 1$. The k_i is the degree of node i. The Γ_i is the neighbor nodes set of node i.

C. 'Loading' modeling and 'capacity' modeling method based on the power product function of node degree and degree sum of its multilevel neighbors.

The model is as follows [19]:

$$L_i = (k_i)^{\alpha_0} \left(\sum_{i_1 \in \Gamma_i} k_{i_1}\right)^{\alpha_1} \left(\sum_{i_2 \in \Gamma_{i_1}} k_{i_2}\right)^{\alpha_2} \cdots \left(\sum_{i_n \in \Gamma_{i_{n-1}}} k_{i_n}\right)^{\alpha_n}, \quad i = 1, \dots, N \tag{5}$$

$$C_i = TL_i, \quad i = 1, \dots, N. \tag{6}$$

In formula (5) and (6), L_i is the loading and C_i is the capacity of node i. The β is a tunable parameter where $\beta > 0$ and the T is a tolerance parameter where $T \geq 1$. The k_i is the degree of node i. $\Gamma_i, \Gamma_{i1}, \dots, \Gamma_{in-1}$ are the first, second, ..., $n-1$ level neighbor nodes set of node i respectively.

Although the model (1)–(6) are relatively good loading and capacity models based on node local information for the unweighted and undirected network, but these models cannot distinguish the loading and capacity differences of large hub nodes and small transit nodes with the same connectivity. In order to make this distinction, we need to build new models of loading-capacity for the directed and weighted network.

For the node weight is the key index that distinguish the differences of the role, influence and throughput of the nodes with the same degree, in this paper, to distinguish the loading and capacity of the nodes with the same degree but with different throughput we will use the loading and capacity modeling method based on weighted node degree.

The model is as follows:

$$L_i^a = \alpha (k_i^{in} w_i^a)^\beta, \quad i = 1, \ldots, N \tag{7}$$

$$C_i^a = TL_i^a, \quad i = 1, \ldots, N. \tag{8}$$

In formula (7) and (8), L_i^a is the loading and C_i^a is the capacity of node i. The β, α are the tunable parameters where β, $\alpha > 0$ and the T is a tolerance parameter where $T \geq 1$. The k_i^{in} is the in-degree of node i and w_i^a is the weight of node i.

3.3. The 'over-loading' cascading failure model of the directed and weighted network

Corresponding with reality condition, we consider that the 'over-loading' flow redistribution of the failure node will take place in its upstream nodes, namely the in-degree set of the failure node. Meanwhile, the 'over-loading' flow redistribution will also be based on the principle of 'the capable ones are the busiest'.

For the physical, costly and technological reasons, every physical system has a limit load capacity for the strength, structure, etc., over which the system will not maintain the normal functions. According to this postulate, we judge whether a network node function properly from the loading view. If the loading cannot satisfy the main measure condition $L_i^a < C_i^a$ $(i = 1, \ldots, N)$, namely if $L_i^a \geq C_i^a$ $(i = 1, \ldots, N)$, this node is failure node and loses efficiency.

When the failure takes place on node i for accidents or deliberate attacks, the loading L_i^a of this node has to be returned to its upstream nodes and diverted to other downstream nodes of these upstream nodes. Based on the principle of 'the capable ones are the busiest', the returned loading L_i^a will be apportioned among the upstream nodes of the failure node according to the following rule. So the increased loading of each upstream node of this failure node is:

$$\Delta L_{ij}^a = L_i^a L_j^a \Big/ \sum_{j' \in \Gamma_i^{in}} L_{j'}^a, \quad j \in \Gamma_i^{in}. \tag{9}$$

The loading of such direct upstream node after flow redistribution is this increased loading plus its original loading.

$$L_j^{a(new)} = L_j^a + \Delta L_{ij}^a = L_j^a + L_i^a L_j^a \Big/ \sum_{j' \in \Gamma_i^{in}} L_{j'}^a, \quad j \in \Gamma_i^{in}. \tag{10}$$

If subsequent failure does not occur on node j, the following condition must be satisfied:

$$L_j^{a(new)} = L_j^a + L_i^a L_j^a \Big/ \sum_{j' \in \Gamma_i^{in}} L_{j'}^a < C_j^a, \quad j \in \Gamma_i^{in}. \tag{11}$$

Because:

$$L_i^a = \alpha (k_i^{in} w_i^a)^\beta, \quad i = 1, \ldots, N \tag{12}$$

$$L_j^a = \alpha (k_j^{in} w_j^a)^\beta, \quad j \in \Gamma_i^{in} \tag{13}$$

$$C_j^a = TL_j^a, \quad j = 1, \ldots, N, T \geq 1. \tag{14}$$

Merging the formula (14) into (11), the following formula can be obtained:

$$L_j^a + L_i^a L_j^a \Big/ \sum_{j' \in \Gamma_i^{in}} L_{j'}^a < TL_j^a, \quad j \in \Gamma_i^{in}. \tag{15}$$

Simplifying above formula:

$$1 + L_i^a \Big/ \sum_{j' \in \Gamma_i^{in}} L_{j'}^a < T. \tag{16}$$

Merging the formulae (12) and (13) into (16), the following formula can be obtained:

$$1 + \alpha (k_i^{in} w_i^a)^\beta \Big/ \sum_{j' \in \Gamma_i^{in}} \alpha (k_{j'}^{in} w_{j'}^a)^\beta < T. \tag{17}$$

Simplifying above formula:

$$1 + (k_i^{in} w_i^a)^\beta \Big/ \sum_{j' \in \Gamma_i^{in}} (k_{j'}^{in} w_{j'}^a)^\beta < T. \tag{18}$$

According to the adjacent relation, node weight probabilistic distribution function of the network and Bayesian formula, we can get:

$$\sum_{j'\in\Gamma_i^{in}}(k_{j'}^{in}w_j^a) = k_i^{in}\sum_{v=1}^{N}[p(k'^{in}_v w'^a_v/k_i^{in}w_i^a)k'^{in}_v w'^a_v]. \tag{19}$$

For the majority of complex networks such as BA, WS, NW, and ER, have degree–degree irrelevant property, the following equation can be obtained:

$$p(k'^{in}_v w'^a_v/k_i^{in}w_i^a) = k'^{in}_v w'^a_v p(k'^{in}_v w'^a_v)/\langle k^{in}w^a\rangle. \tag{20}$$

Consequently, we get the following equations:

$$\begin{aligned}\sum_{j'\in\Gamma_i^{in}}(k_{j'}^{in}w_{j'}^a) &= k_i^{in}\sum_{v=1}^{N}[k'^{in}_v w'^a_v p(k'^{in}_v w'^a_v)k'^{in}_v w'^a_v/\langle k^{in}w^a\rangle]\\ &= k_i^{in}\sum_{v=1}^{N}[(k'^{in}_v w'^a_v)^2 p(k'^{in}_v w'^a_v)/\langle k^{in}w^a\rangle]\\ &= k_i^{in}\langle (k^{in}w^a)^2\rangle/\langle k^{in}w^a\rangle\\ &= k_i^{in}\langle (k^{in})^2\rangle\,\langle (w^a)^2\rangle/(\langle k^{in}\rangle\,\langle w^a\rangle)\end{aligned} \tag{21}$$

$$\begin{aligned}\sum_{j'\in\Gamma_i^{in}}(k_{j'}^{in}w_{j'}^a)^\beta &= k_i^{in}\langle (k^{in}w^a)^{\beta+1}\rangle/\langle k^{in}w^a\rangle\\ &= k_i^{in}\langle (k^{in})^{\beta+1}\rangle\,\langle (w^a)^{\beta+1}\rangle/(\langle k^{in}\rangle\langle w^a\rangle).\end{aligned} \tag{22}$$

Merging the formulae (22) into (18), we get:

$$\begin{aligned}1+(k_i^{in}w_i^a)^\beta\Big/\sum_{j'\in\Gamma_i^{in}}(k_{j'}^{in}w_{j'}^a)^\beta &= 1+(k_i^{in}w_i^a)^\beta/[k_i^{in}\langle (k^{in})^{\beta+1}\rangle\,\langle (w^a)^{\beta+1}\rangle/(\langle k^{in}\rangle\,\langle w^a\rangle)]\\ &= 1+\frac{(k_i^{in})^{\beta-1}(w_i^a)^\beta\langle k^{in}\rangle\,\langle w^a\rangle}{\langle (k^{in})^{\beta+1}\rangle\,\langle (w^a)^{\beta+1}\rangle} < T.\end{aligned} \tag{23}$$

Thus the critical threshold T_C of the 'over-loading' cascading failure is:

$$T_c = \begin{cases} 1+\dfrac{(k_{\min}^{in})^{\beta-1}\langle k^{in}\rangle\,\langle w^a\rangle(w_{\max}^a)^\beta}{\langle (k^{in})^{\beta+1}\rangle\,\langle (w^a)^{\beta+1}\rangle}, & 0<\beta<1\\ 1+\dfrac{\langle k^{in}\rangle\,\langle w^a\rangle w_{\max}^a}{\langle (k^{in})^2\rangle\,\langle (w^a)^2\rangle}, & \beta=1\\ 1+\dfrac{(k_{\max}^{in})^{\beta-1}\langle k^{in}\rangle\,\langle w^a\rangle(w_{\max}^a)^\beta}{\langle (k^{in})^{\beta+1}\rangle\,\langle (w^a)^{\beta+1}\rangle}, & \beta>1.\end{cases} \tag{24}$$

From formula (24) we can see that the over-loading cascading failures are related to the in-degree and the node weight of the failure node. The more homogenized the network's node in-degrees and node weight probabilistic distributions are, the smaller the critical threshold T_c is, and at the same conditions, the stronger the network's resistibility of cascading failures is, the lower the network's construction cost is. Furthermore, the higher the in-degree average of the network is, the stronger the network's resistibility of 'over-loading' cascading failures is at the same conditions. The robustness of the network is the highest when $\beta = 1$.

3.4. The 'short-loading' cascading failure model of the directed and weighted network

After a node failure takes place, different from the increase loading of the upstream nodes who are compelled to share in the loading of the failure node, the downstream nodes will not receive the flow from the failure node again. It will cause the downstream nodes 'flow shortage' of the deserved material, energy and information resources (such as the electric supply of electric power network, and oil transportation of oil pipe network), even a mass of downstream nodes cascading reaction for 'short-loading' at extreme situations, which engenders cascading failure as a result of 'short-loading'.

For this class of cascading failure modeling, on account of occurring on the downstream nodes and having 'supply shortage' property, the 'short-loading' cascading failure model is different from the 'over-loading' cascading failure model (24) in the III. C. of this paper. The flow intermitted from the failure node results in that the deserved loading of the direct downstream nodes are reduced. The deserved loading of the direct downstream nodes will be reduced according to requirement based on the loading ability of each direct downstream node. The reduced loading of the direct downstream node q is:

$$\Delta L_{iq}^{a} = L_i^a L_q^a \Big/ \sum_{q' \in \Gamma_i^{out}} L_{q'}^a, \quad q \in \Gamma_i^{out}. \tag{25}$$

The loading of the direct downstream node q after flow redistribution for loading decrease is that the deserved loading subtracts above reduced loading:

$$L_q^{a(new)} = L_q^a - \Delta L_{ij}^a = L_q^a - L_i^a L_q^a \Big/ \sum_{q' \in \Gamma_i^{out}} L_{q'}^a, \quad q \in \Gamma_i^{out}. \tag{26}$$

If the successive failure for 'supply shortage' does not occur at the node q, the following condition should be satisfied:

$$L_q^{a(new)} = L_q^a - L_i^a L_q^a \Big/ \sum_{q' \in \Gamma_i^{out}} L_{q'}^a \geq C_w^a, \quad q \in \Gamma_i^{out} \tag{27}$$

where C_w^a is the minimum required supply loading (material, energy, information, etc. resources) to ensure the normal efficiency of the node.

Using the derivation method and procedure similar to formula (15), (18) and (23), we can get:

$$L_q^a - L_i^a L_q^a \Big/ \sum_{q' \in \Gamma_i^{out}} L_{q'}^a \geq T_w L_w^a, \quad q \in \Gamma_i^{out} \tag{28}$$

$$1 - (k_i^{in} w_i^a)^\beta \Big/ \sum_{q' \in \Gamma_i^{out}} (k_{q'}^{in} w_{q'}^a)^\beta \geq T_w \tag{29}$$

$$1 - \frac{(k_i^{in})^\beta (w_i^a)^\beta \langle k^{in} \rangle \langle w^a \rangle}{(k_i^{out}) \langle (k^{in})^{\beta+1} \rangle \langle (w^a)^{\beta+1} \rangle} \geq T_w. \tag{30}$$

Thus the critical threshold T_w of the 'short-loading' cascading failure is:

$$T_w = 1 - \frac{(k_{\max}^{in})^\beta \langle k^{in} \rangle \langle w^a \rangle (w_{\max}^a)^\beta}{k_{\min}^{out} \langle (k^{in})^{\beta+1} \rangle \langle (w^a)^{\beta+1} \rangle}, \quad \beta > 0. \tag{31}$$

From formula (31) we can see that the short-loading cascading failures are related to the out-degree, in-degree and node weight of the failure node. For 'short-loading' cascading failure, the maximum in-degree, average in-degree, minimum out-degree and maximum node weight of the failure node have great impact on the robustness of the network. For a network, the smaller the maximum in-degree is and the greater the minimum out-degree is, the greater the critical threshold T_w is, and the stronger the endurance to 'short-loading' cascading failure. Secondly, the in-degree and node weight probabilistic distributions of the network affect 'short-loading' cascading failure greatly. The more approaching Poisson distribution or exponential distribution the in-degree and node weight probabilistic distributions are, the larger the critical threshold T_w is, and at the same conditions, the stronger the network's resistibility of 'short-loading' cascading failures is, the lower the network's construction cost is.

4. Cascading failure vulnerability analysis of two typical directed and weighted networks

For real-world networks, from the perspective of connectivity, nothing less than the two class: homogeneous connectivity network and heterogeneous connectivity network. The random network with Poisson distribution can be used to simulate the former. The scale-free network with power-law distribution can be used to describe the latter. The actual examples of these two classes of networks are American electricity network which is the representative of the network whose node degree distribution is Poisson distribution [30,31], and Internet, the World Wide Web which are examples of the network whose node degree distribution is power-law distribution [32,33]. So, we will take two typical real-world networks – power law network and random network – as an example to analyze the network resistibility for 'over-loading' and 'short-loading' cascading failures. The corresponding probability distributions are the power-law distribution and the Poisson distribution respectively.

To make the model practical significance, here $0 \leq \beta \leq 3$ choosing integer is supposed for Poisson distribution, and $0 < \beta < 1$ is supposed for power law distribution.

4.1. 'Over-loading' and 'short-loading' cascading failure resistibility analysis of the network whose in-degree, out-degree and node weight with Poisson distribution

Here we use the method of generating functions to derive the threshold function of Poisson distribution network cascading failure, and build explicit functional relationship between the critical threshold of cascading failure and the probabilistic distributions of in-degree, out-degree and node weight.

For the probabilistic distribution function of the Poisson distribution is $p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, the corresponding probability generating function is $g(x) = e^{\lambda(x-1)}$. So the first-order moment and the $\beta + 1$ order moment can be obtained from the probability generating function calculation when $1 \le \beta \le 3$.

$$\langle x\rangle = g'(x)|_{x=1} = \lambda e^{\lambda(x-1)}|_{x=1} = \lambda \tag{32}$$

$$\langle x^{\beta+1}\rangle = g''(x)|_{x=1} + g'(x)|_{x=1} = \lambda^2 + \lambda, \quad \beta = 1 \tag{33}$$

$$\langle x^{\beta+1}\rangle = g'''(x)|_{x=1} + 3g''(x)|_{x=1} + g'(x)|_{x=1} = \lambda^3 + 3\lambda^2 + \lambda, \quad \beta = 2 \tag{34}$$

$$\begin{aligned}\langle x^{\beta+1}\rangle &= g^{(4)}(x)|_{x=1} + 6g'''(x)|_{x=1} + 7g''(x)|_{x=1} + g'(x)|_{x=1}\\ &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda, \quad \beta = 3.\end{aligned} \tag{35}$$

If the in-degree, out-degree and node weight of a network obey Poisson distribution, we postulate that their λ parameters are respectively equal to λ^{in}, λ^{out} and λ^{w}. So the critical threshold T_c^p of 'over-loading' cascading failure for this network is calculated by the following formulae:

$$T_c^p = \begin{cases} 1 + \dfrac{w_{\max}^a}{(\lambda^{in}+1)(\lambda^w+1)}, & \beta = 1\\ 1 + \dfrac{k_{\max}^{in}(w_{\max}^a)^2}{[(\lambda^{in})^2 + 3\lambda^{in} + 1][(\lambda^w)^2 + 3\lambda^w + 1]}, & \beta = 2\\ 1 + \dfrac{(k_{\max}^{in})^2(w_{\max}^a)^3}{[(\lambda^{in})^3 + 6(\lambda^{in})^2 + 7\lambda^{in} + 1][(\lambda^w)^3 + 6(\lambda^w)^2 + 7\lambda^w + 1]}, & \beta = 3.\end{cases} \tag{36}$$

The network's critical threshold T_w^p of 'short-loading' cascading failure is calculated by the following formulae:

$$T_w^p = \begin{cases} 1 - \dfrac{k_{\max}^{in} w_{\max}^a}{k_{\min}^{out}(\lambda^{in}+1)(\lambda^w+1)}, & \beta = 1\\ 1 - \dfrac{(k_{\max}^{in})^2(w_{\max}^a)^2}{k_{\min}^{out}[(\lambda^{in})^2 + 3\lambda^{in} + 1][(\lambda^w)^2 + 3\lambda^w + 1]}, & \beta = 2\\ 1 - \dfrac{(k_{\max}^{in})^3(w_{\max}^a)^3}{k_{\min}^{out}[(\lambda^{in})^3 + 6(\lambda^{in})^2 + 7\lambda^{in} + 1][(\lambda^w)^3 + 6(\lambda^w)^2 + 7\lambda^w + 1]}, & \beta = 3.\end{cases} \tag{37}$$

Here we use numerical simulation method, calculated the quantitative relationship surface charts between 'over-loading' threshold T_{cp}, 'short-loading' threshold T_{wp} and node in-degree distribution parameter λ^{in}, node weight distribution parameter λ^{w} respectively for Poisson distribution network.

When $\beta = 1, \beta = 2, \beta = 3$, for Poisson distribution network, the surface charts of 'over-loading' threshold T_c^p with the node in-degree and node weight distribution parameters λ^{in}, λ^{w} are shown in Figs. 3–5 respectively. The corresponding contour graphics are shown in Figs. 6–8 respectively.

When $\beta = 1, \beta = 2, \beta = 3$, for Poisson distribution network, the surface charts of 'short-loading' threshold T_w^p with the node out-degree and node weight distribution parameters λ^{out}, λ^{w} are shown in Figs. 9–11 respectively. The corresponding contour graphics are shown in Figs. 12–14 respectively.

4.2. 'Over-loading' and 'short-loading' cascading failure resistibility analysis of the network whose in-degree, out-degree and node weight with power law distribution

For the probabilistic distribution function of the power law distribution is $p(x) = cx^{-\gamma}(\gamma > 1), c \approx (\gamma-1)x_{\min}^{\gamma-1}(\gamma > 1)$. When $0 < \beta < 1$, the $\langle x\rangle$ and $\langle x^{\beta+1}\rangle$ can be calculated from probabilistic function $p(x)$. To make the problem practical significance, $\gamma > 2$ is required, which corresponds to that the power exponent of most power law networks is located in the interval $2 < \gamma < 3$.

If x obeys the power law distribution $p(x) = cx^{-\gamma}(\gamma > 1)$, the formula $x_{\max} \approx x_{\min}N^{\frac{1}{\gamma-1}}(\gamma > 1)$ can be derived from the probability statistic theory, where N is the total number of discrete magnitude x. Thus we get:

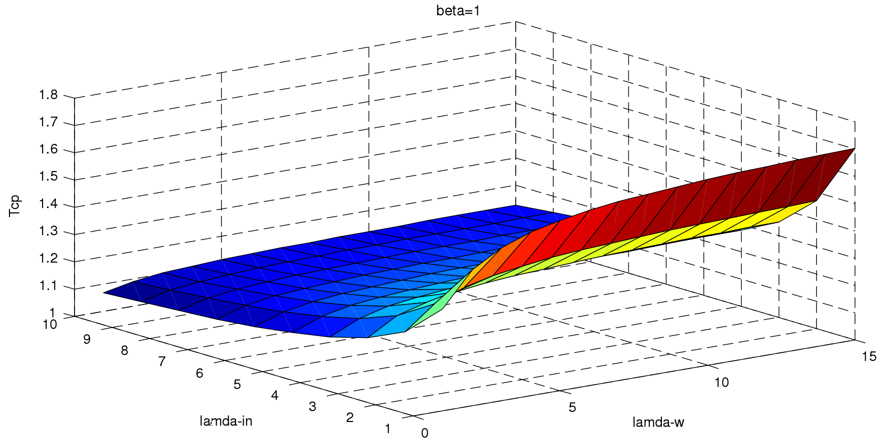

Fig. 3. The surface of 'over-loading' threshold T_c^p with the nodes average in-degree λ^{in} and average nodes weight λ^{w} for Poisson distribution network when the control parameter β=1.

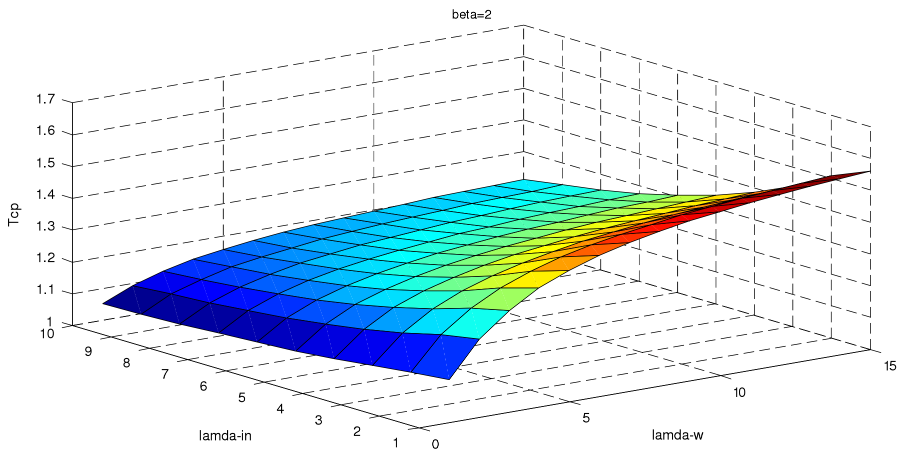

Fig. 4. The surface of 'over-loading' threshold T_c^p with the nodes average in-degree λ^{in} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 2$.

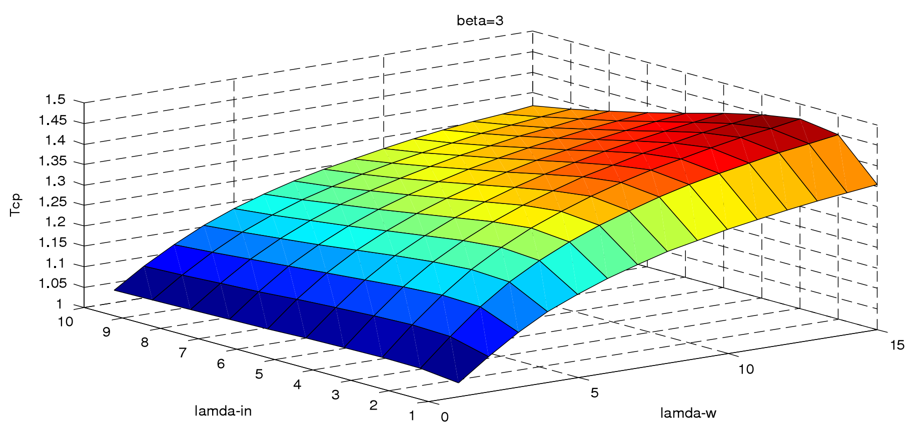

Fig. 5. The surface of 'over-loading' threshold T_c^p with the nodes average in-degree λ^{in} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 3$.

$$\begin{aligned}\langle x\rangle &= c\frac{1}{-\gamma+2}x^{-\gamma+2}|_{x=x_{\min}}^{x=x_{\max}}\\ &= c\frac{1}{-\gamma+2}(x_{\max}^{-\gamma+2}-x_{\min}^{-\gamma+2})\\ &= c\frac{1}{-\gamma+2}x_{\min}^{-\gamma+2}\left(N^{-\frac{\gamma-2}{\gamma-1}}-1\right)\quad(\gamma>2)\end{aligned}\tag{38}$$

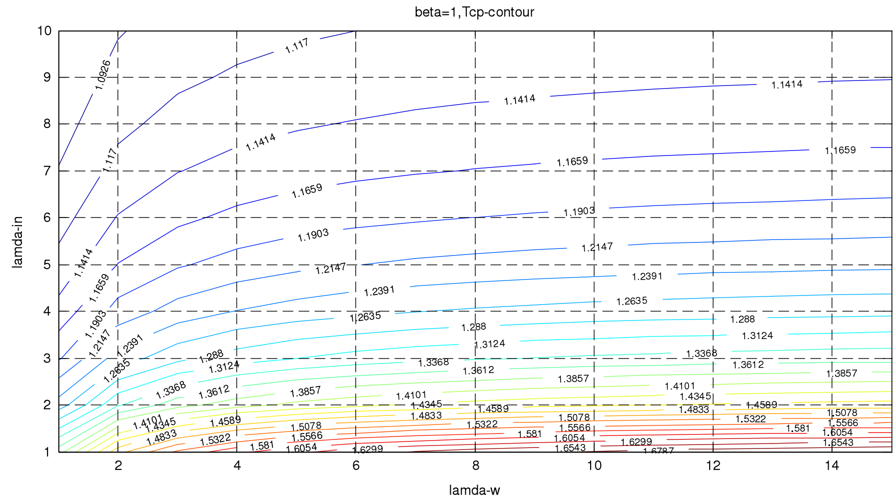

Fig. 6. The contour lines of 'over-loading' threshold T_c^p with the nodes average in-degree λ^{in} and average nodes weight λ^w for Poisson distribution network when the control parameter β=1.

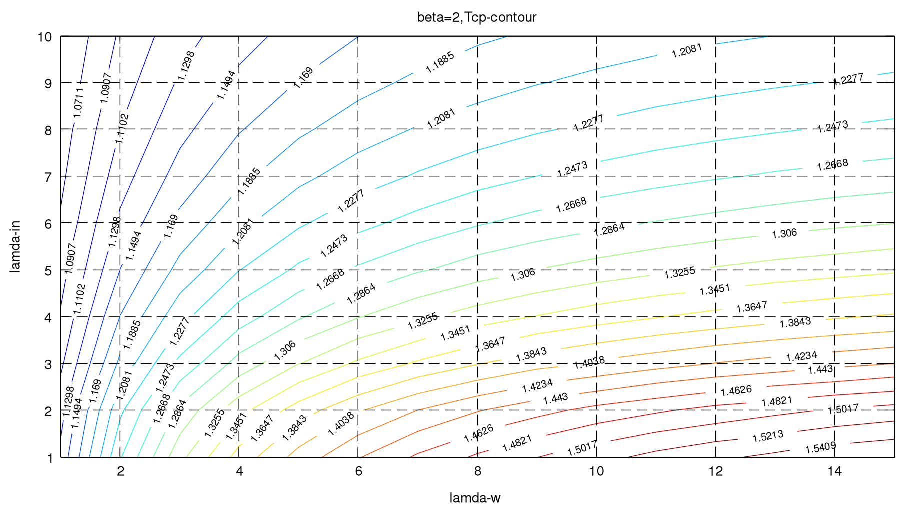

Fig. 7. The contour lines of 'over-loading' threshold T_c^p with the nodes average in-degree λ^{in} and average nodes weight λ^w for Poisson distribution network when the control parameter $\beta = 2$.

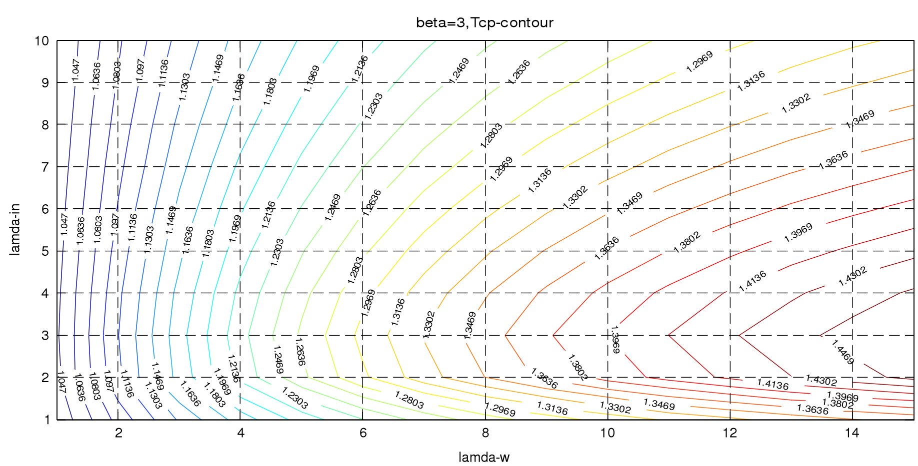

Fig. 8. The contour lines of 'over-loading' threshold T_c^p with the nodes average in-degree λ^{in} and average nodes weight λ^w for Poisson distribution network when the control parameter $\beta = 3$.

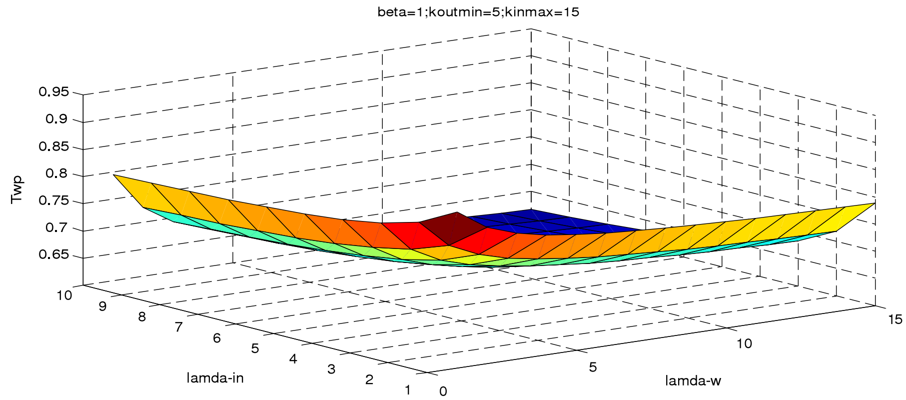

Fig. 9. The surface of 'short-loading' threshold T_w^p with the nodes average out-degree λ^{out} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 1$.

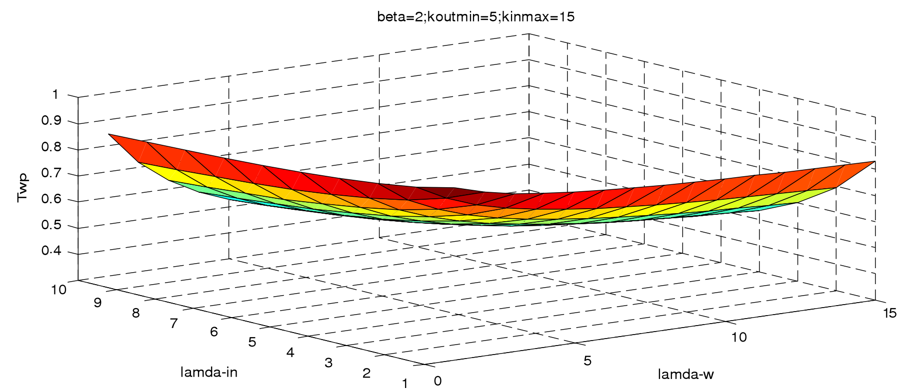

Fig. 10. The surface of 'short-loading' threshold T_w^p with the nodes average out-degree λ^{out} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 2$.

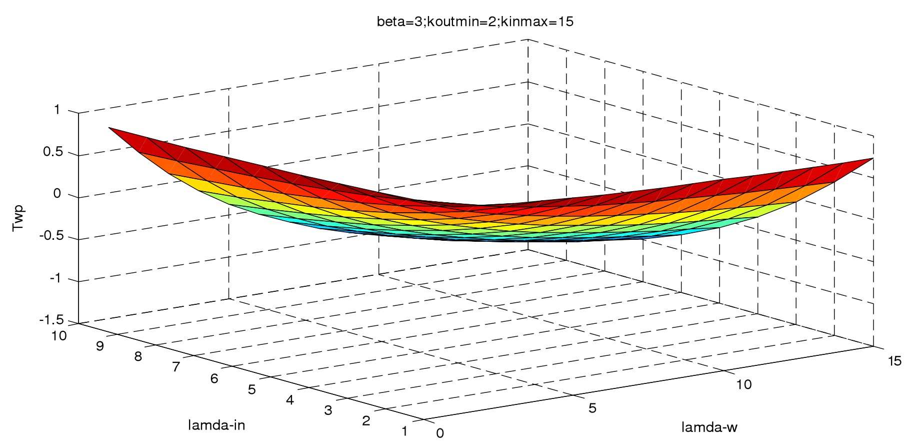

Fig. 11. The surface of 'short-loading' threshold T_w^p with the nodes average out-degree λ^{out} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 3$.

$$\begin{aligned}
\langle x^{\beta+1}\rangle &= c\frac{1}{-\gamma+\beta+2}x^{-\gamma+\beta+2}|_{x=x_{\min}}^{x=x_{\max}} \\
&= c\frac{1}{-\gamma+\beta+2}(x_{\max}^{-\gamma+\beta+2} - x_{\min}^{-\gamma+\beta+2}) \\
&= c\frac{1}{\gamma-2-\beta}x_{\min}^{-\gamma+\beta+2}\left(1 - N^{-\frac{\gamma-2-\beta}{\gamma-1}}\right) \quad (\gamma > 2,\ 0 < \beta < \gamma - 2).
\end{aligned} \tag{39}$$

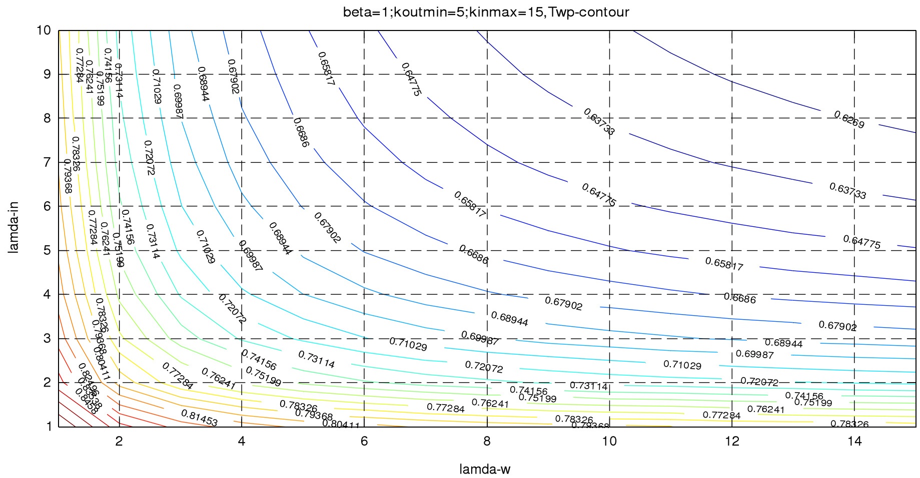

Fig. 12. The contour lines of 'short-loading' threshold T_w^p with the nodes average out-degree λ^{out} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 1$.

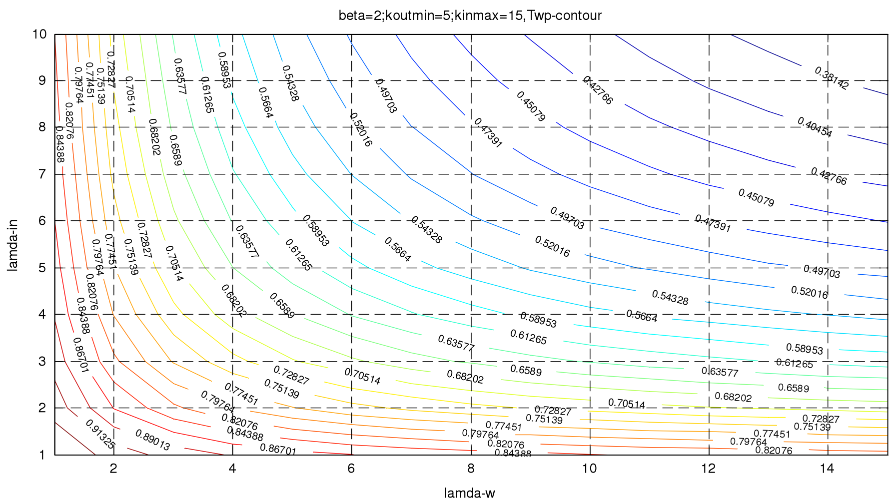

Fig. 13. The contour lines of 'short-loading' threshold T_w^p with the nodes average out-degree λ^{out} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 2$.

So if the in-degree, out-degree and node weight of a network obey power law distribution, and its scaling exponent satisfies $\gamma > 2$, the critical threshold T_c^{sf} of 'over-loading' cascading failure for this network is calculated by the following formulae:

$$\begin{aligned}
\mathrm{T_c^{sf}} &= 1 + \frac{(k_{\min}^{in})^{\beta-1} < k^{in} > \langle w^a \rangle (w_{\max}^a)^{\beta}}{\langle (k^{in})^{\beta+1} \rangle \, \langle (w^a)^{\beta+1} \rangle} \\
&= 1 + \frac{(k_{\min}^{in})^{-1}(-\gamma^{in}+\beta+2)(-\gamma^{w}+\beta+2)}{(-\gamma^{in}+2)(-\gamma^{w}+2)} \times \frac{1 - N^{\frac{-\gamma^{in}+2}{\gamma^{in}-1}}}{1 - N^{\frac{-\gamma^{in}+\beta+2}{\gamma^{in}-1}}} \times \frac{1 - N^{\frac{\gamma^{w}-2}{\gamma^{w}-1}}}{1 - N^{\frac{\gamma^{w}-\beta-2}{\gamma^{w}-1}}}, \\
& 0 < \beta \le \min(\gamma^{in}, \gamma^{w}) - 2. \qquad (40)
\end{aligned}$$

When the N is great, the following formula is workable.

$$\mathrm{T_c^{sf}} = 1 + (k_{\min}^{in})^{-1}\left(1 - \frac{\beta}{\gamma^{in}-2}\right)\left(1 - \frac{\beta}{\gamma^{w}-2}\right), \quad 0 < \beta \le \min(\gamma^{in}, \gamma^{w}) - 2. \qquad (41)$$

When the N is great, similarly available the critical threshold T_w^{sf} of 'short-loading' cascading failure for this network is:

$$\mathrm{T_w^{sf}} = 1 - (k_{\min}^{out})^{-1}\left(1 - \frac{\beta}{\gamma^{in}-2}\right)\left(1 - \frac{\beta}{\gamma^{w}-2}\right), \quad 0 < \beta \le \min(\gamma^{out}, \gamma^{w}) - 2. \qquad (42)$$

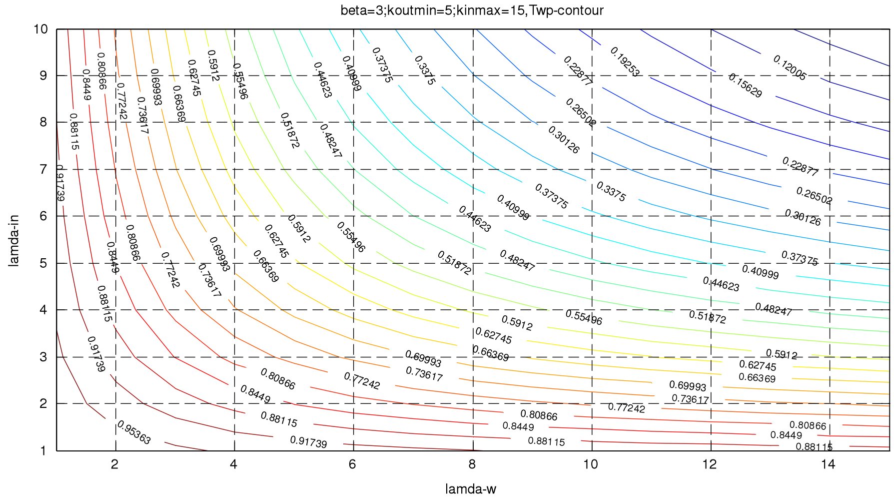

Fig. 14. The contour lines of 'short-loading' threshold T_w^p with the nodes average out-degree λ^{out} and average nodes weight λ^{w} for Poisson distribution network when the control parameter $\beta = 3$.

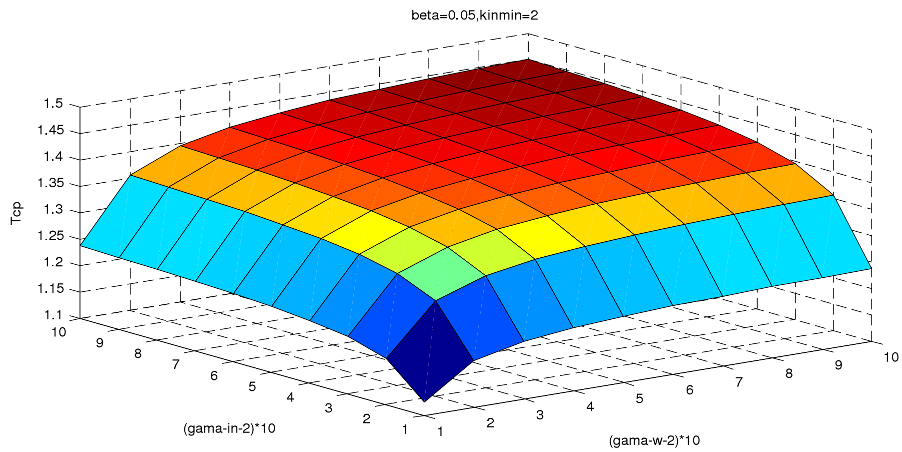

Fig. 15. The surface of 'over-loading' threshold T_c^p with the in-degree scale γ^{in} and node weight scale γ^w for SF (Scale Free) network when the control parameter $\beta = 0.05$ and the minimum in-degree $k_{\mathrm{min}}^{\mathrm{in}} = 2$.

Here we can also use numerical simulation method, calculated the quantitative relationship surface charts between 'over-loading' threshold T_{cp}, 'short-loading' threshold T_{wp} and node in-degree distribution parameter γ^{in}, node weight distribution parameter γ^w respectively for power law distribution network.

When $\beta = 0.05$ and $k_{\mathrm{min}}^{in} = 2$, $k_{\mathrm{min}}^{in} = 5$, $k_{\mathrm{min}}^{in} = 15$, for power law distribution network, the surface charts of 'over-loading' threshold T_c^p with the node in-degree scales γ^{in} and node weight scales γ^w are shown in Figs. 15–17 respectively. The corresponding contour graphics are shown in Figs. 18–20 respectively.

When $\beta = 0.05$ and $k_{\mathrm{min}}^{out} = 2$, $k_{\mathrm{min}}^{out} = 5$, $k_{\mathrm{min}}^{out} = 15$, for power law distribution network, the surface charts of 'short-loading' threshold T_w^p with the node out-degree scales γ^{out} and node weight scales γ^w are shown in Figs. 21–23 respectively. The corresponding contour graphics are shown in Figs. 24–26 respectively.

5. Simulation experiment and discovery

To confirm our ideas about cascading failure model of directed and weighted networks, on the basis of the above analysis, we have also developed the simulation program which can be used to study the cascading failure process of the directed and weighted networks and discover their laws of the cascading failure occurrence. By the way of the some simulation experiments, we have a few of new discoveries for the cascading failure process and law of the directed and weighted networks. Firstly, the short-load failure can bring the same huge disastrous results as the overload failure. This discovery can be also proved by the largest blackout in US history took place on 14 August 2003 and the Western North American blackouts in July and August 1996. Secondly, for the ER network (the example of network with Poisson distribution) and BA network (the example of network with power distribution), the two Tc threshold points of the overload cascading failure

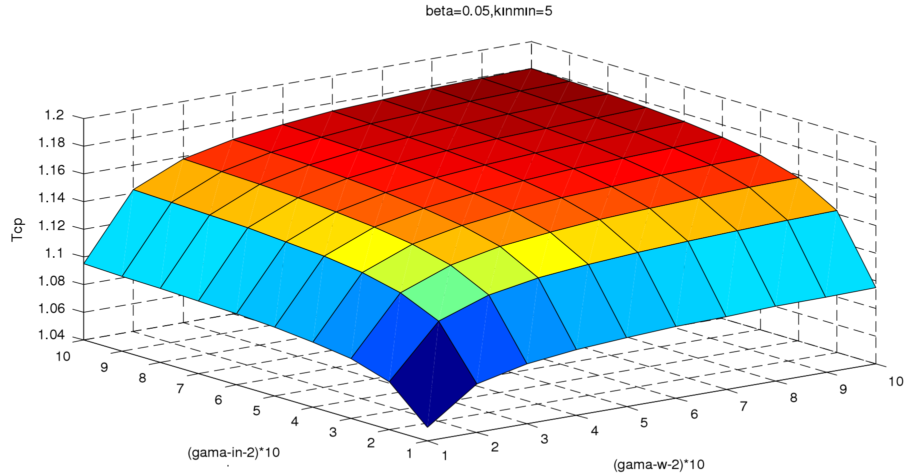

Fig. 16. The surface of 'over-loading' threshold T_c^p with the in-degree scale γ^{in} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum in-degree $k_{\min}^{in} = 5$.

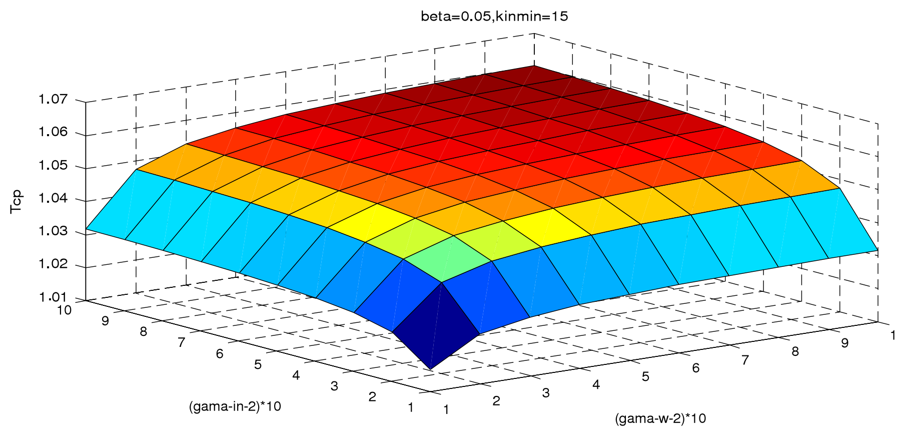

Fig. 17. The surface of 'over-loading' threshold T_c^p with the in-degree scale γ^{in} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum in-degree $k_{\min}^{in} = 15$.

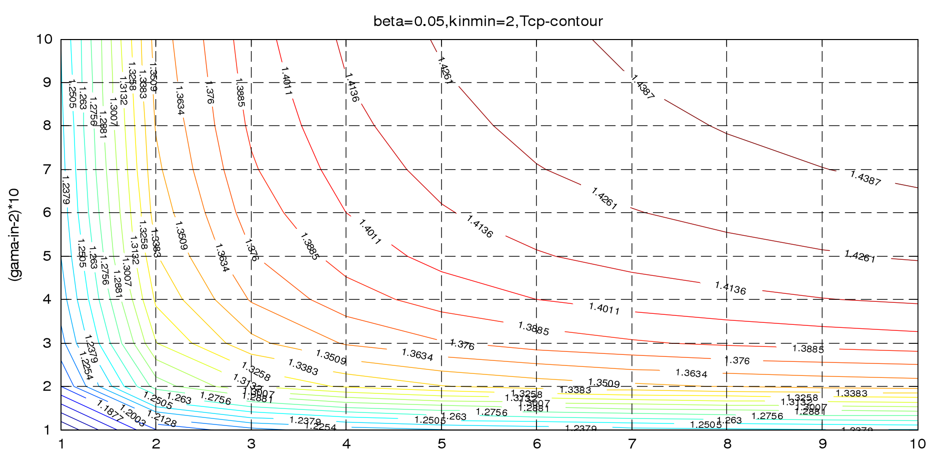

Fig. 18. The contour lines of 'over-loading' threshold T_c^p with the in-degree scale γ^{in} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum in-degree $k_{\min}^{in} = 2$.

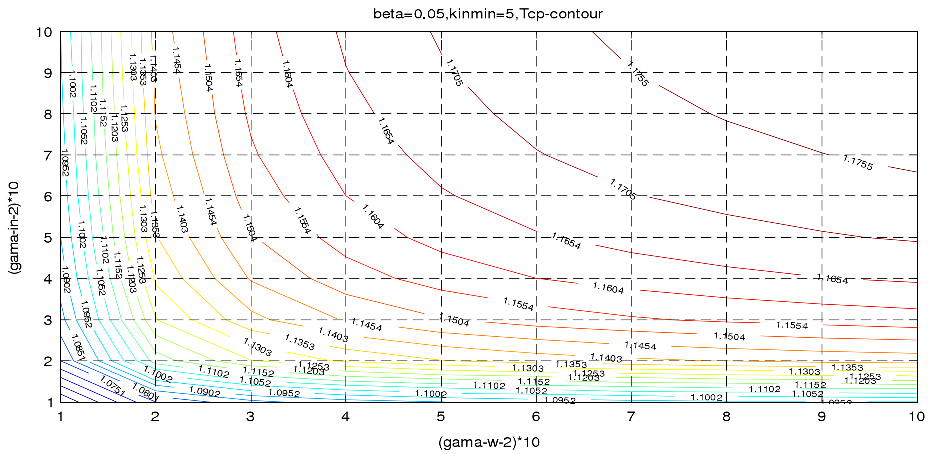

Fig. 19. The contour lines of 'over-loading' threshold T_c^p with the in-degree scale γ^{in} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum in-degree $k_{\min}^{in} = 5$.

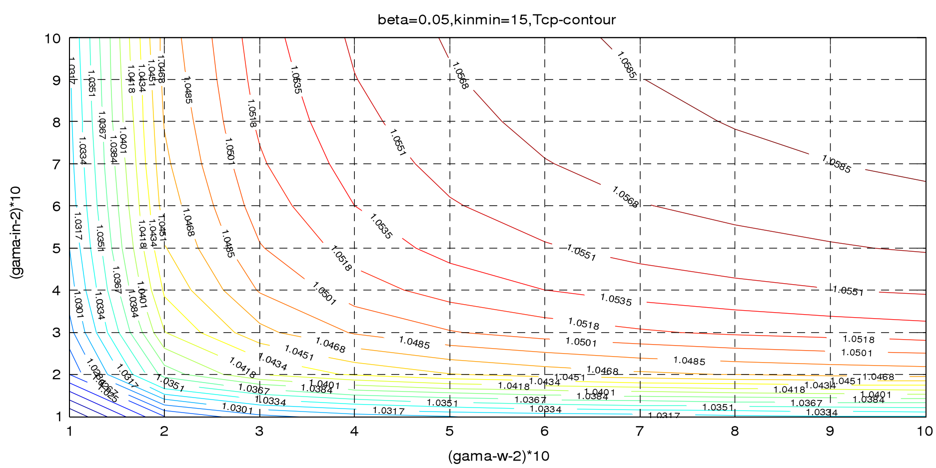

Fig. 20. The contour lines of 'over-loading' threshold T_c^p with the in-degree scale γ^{in} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum in-degree $k_{\min}^{in} = 15$.

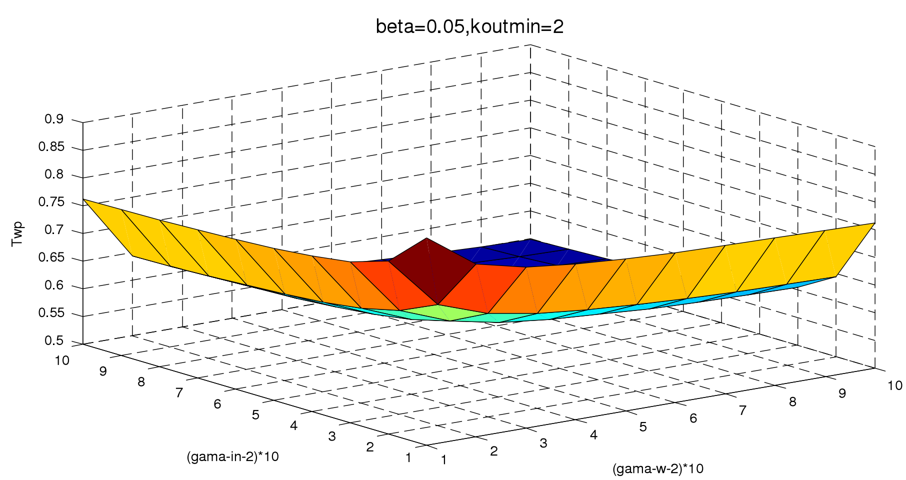

Fig. 21. The surface of 'short-loading' threshold T_w^p with the out-degree scale γ^{out} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum out-degree $k_{\min}^{\text{out}} = 2$.

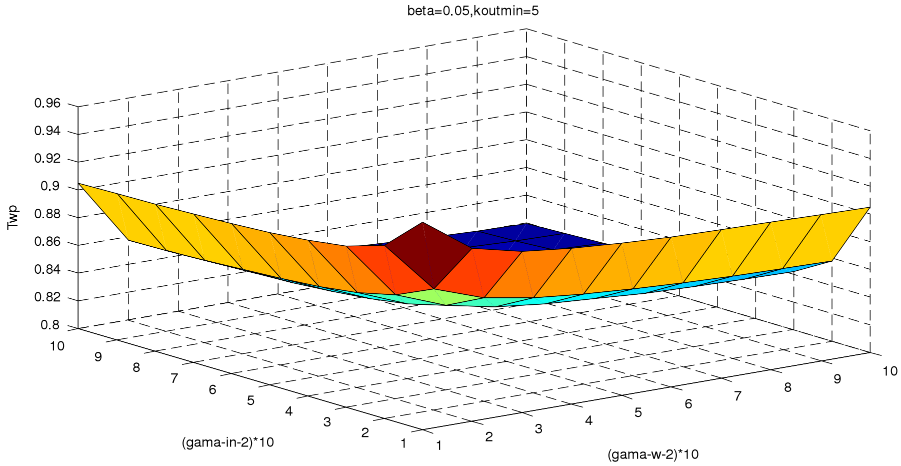

Fig. 22. The surface of 'short-loading' threshold T_w^p with the out-degree scale γ^{out} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum out-degree $k_{min}^{out} = 5$.

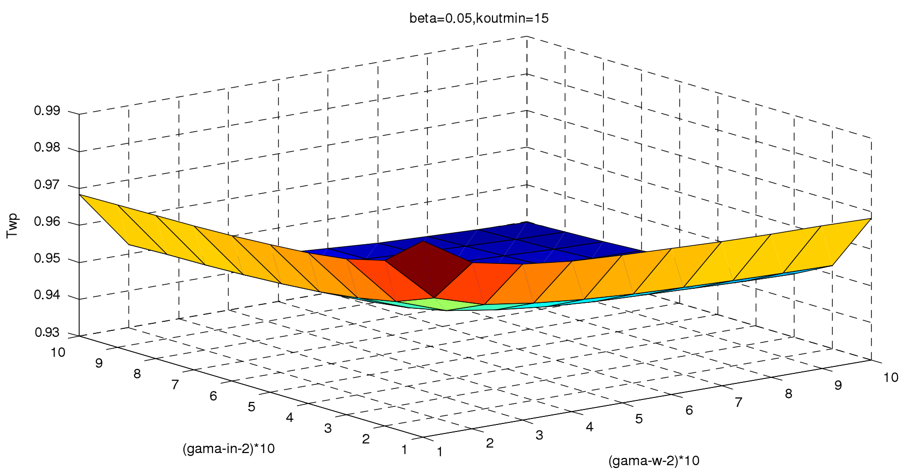

Fig. 23. The surface of 'short-loading' threshold T_w^p with the out-degree scale γ^{out} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum out-degree $k_{min}^{out} = 15$.

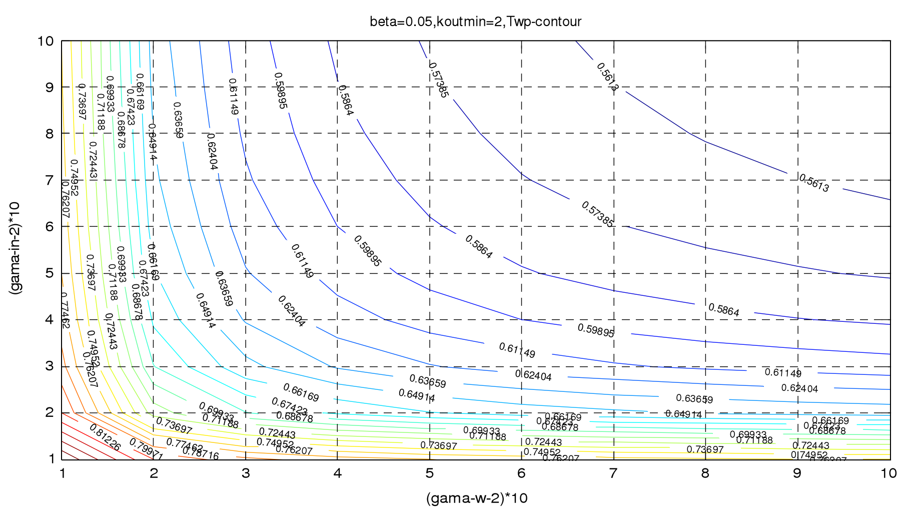

Fig. 24. The contour liners of 'short-loading' threshold T_w^p with the out-degree scale γ^{out} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum out-degree $k_{min}^{out} = 2$.

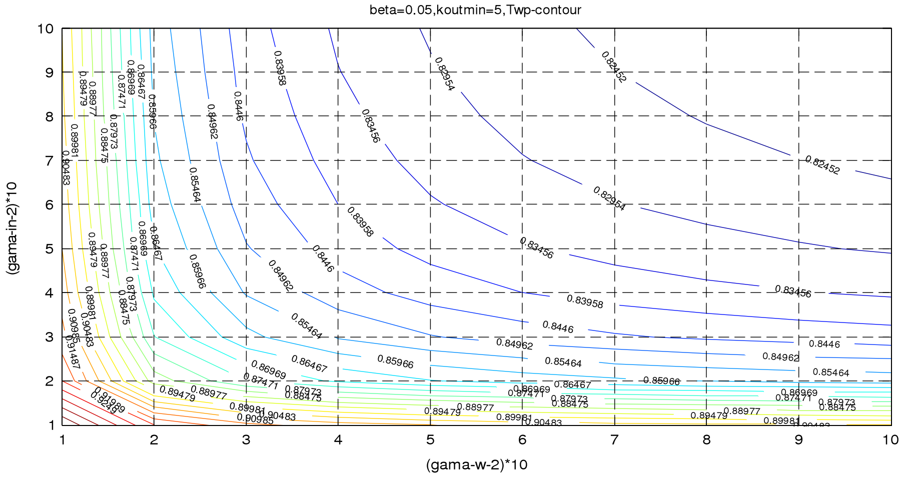

Fig. 25. The contour liners of 'short-loading' threshold T_w^p with the out-degree scale γ^{out} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum out-degree $k_{min}^{out} = 5$.

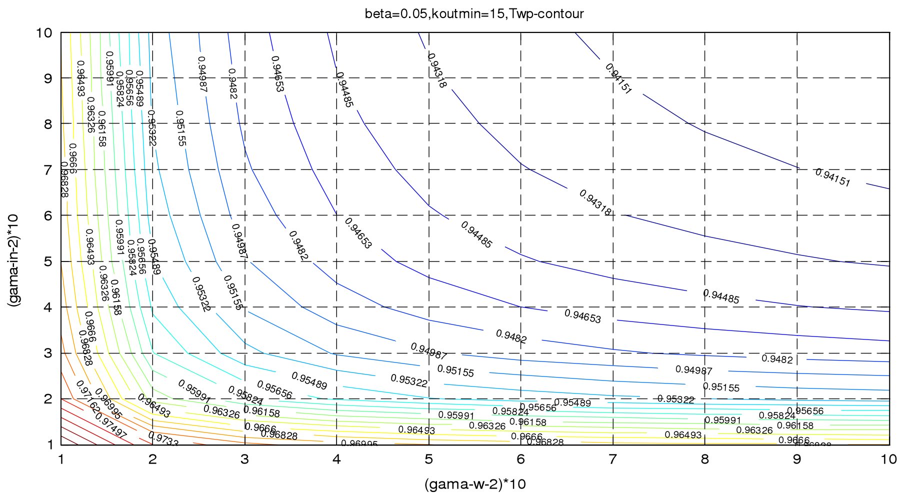

Fig. 26. The contour liners of 'short-loading' threshold T_w^p with the out-degree scale γ^{out} and node weight scale γ^w for SF network when the control parameter $\beta = 0.05$ and the minimum out-degree $k_{min}^{out} = 15$.

of the directed network have be found. The low threshold point T_{c1} is found in the [1.0001, 1.01] (it is discovered in a high-resolution simulation mode). The high threshold point T_{c2} is found respectively in the vicinal value 1.1, 1.2 or 1.3 (its exact value would be decided by the β value). When $1 \le T \le T_{c1}$ and the cascading failure occurred, almost entire network would be paralyzed; however, when $T_{c1} < T < T_{c2}$ and the cascading failure occurred, only 30%–40% nodes would be impacted. Moreover, one and only point T_w of the shortload cascading failure is found in the vicinal value 0.9 or 0.65 (its exact value would be also decided by the β value); but when $T_w \le T \le 1$, it is possible to happen to bring the entire network paralyzed (this result would mainly depend on what failure node is selected). Lots of the real-world examples can also support this result.

The above conclusions are gained by a series of the following simulation experiments.

Here, N=1000 (the number of nodes in the ER network or the BA network). For the ER network, $\beta = 0.1, 0.5, 1.0, 2.0, 3.0$. For the BA network, $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$. After the cascading process is over, we will calculate the number of broken nodes. To this end, we use N_{fn}^c, N_{fn}^w and N_{fn}^s respectively to denote the overload, shortload and total avalanche size induced by an accidental failure. For $0 \le N_{fn}^c \le N, 0 \le N_{fn}^w \le N, 0 \le N_{fn}^s \le N$, when we use R_{Nfn}^c, R_{Nfn}^w and R_{Nfn}^s respectively to denote the overload, shortload and total avalanche rate, it is obvious that the formulae $0 \le R_{Nfn}^c = N_{fn}^c/N \le 1$, $0 \le R_{Nfn}^w = N_{fn}^w/N \le 1$ and $0 \le R_{Nfn}^s = N_{fn}^s/N \le 1$ can be gained.

5.1. The simulation experiment results of the overload and shortload cascading failure of the ER networks

A. The simulation experiment results of the overload cascading failure of the ER networks in which both the shortload and the weights had not be considered.

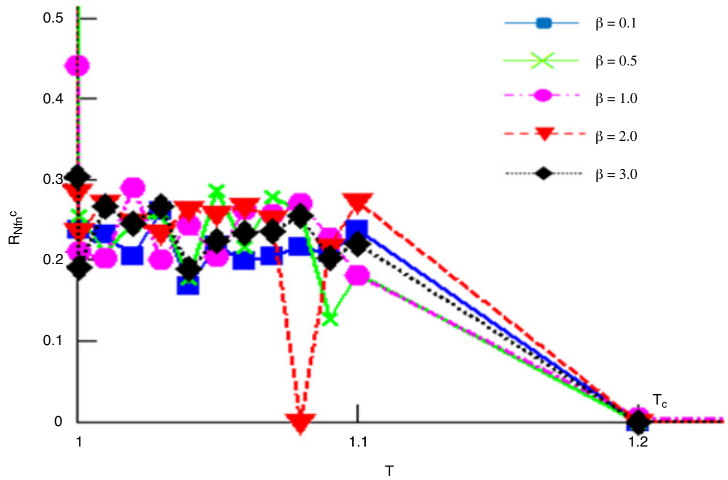

Fig. 27. The results of the overload cascading failure of the ER networks without weights and shortload after the accidental failure.

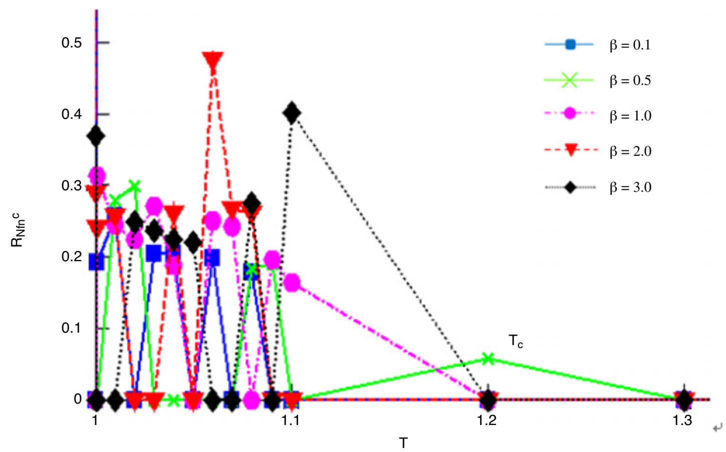

Fig. 28. The results of the overload cascading failure of the ER networks in which the weights had not be considered, but the shortload had be considered, after the accidental failure.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R_{Nfn}^{c} (it is shown in Fig. 27).

Based on Fig. 27, we can gain the following result. For the ER network in which the shortload and the weights had not be considered, when $T_{overload} > 1.01$ (low cost), 70% of the network nodes can be kept from being damaged by the overload failure after the accidental failure. When $T_{overload} > 1.2$ (high cost), almost all the network nodes can be kept from being damaged by the overload after the accidental failure.

B. The simulation experiment results of the overload cascading failure of the ER networks in which the weights had not be considered, but the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R_{Nfn}^{c} (it is shown in Fig. 27).

Based on Fig. 28, we can gain the following result. For the ER network in which the weights had not be considered, but the shortload had be considered, when $T_{overload} > 1.01$ and $T_{shortload} < 0.9$ (still low cost), only 50% of the network nodes can be kept from being damaged by the overload and shortload failure after the accidental failure. When $T_{overload} > 1.3$ and $T_{shortload} < 0.9$ (higher cost), almost all the network nodes can be kept from being damaged by the overload after the accidental failure.

C. The simulation experiment results of the shortload cascading failure of the ER networks in which the weights had not be considered, but the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R_{Nfn}^{w} (it is shown in Fig. 29).

Obviously, Fig. 29 is greatly different from Figs. 28 and 27 in the curve characteristic. When $0.9 < T_{shortload} < 1$, the entire network would possibly be paralyzed. If the 'upstream' node failure happens to occur, it is possible to bring all nodes

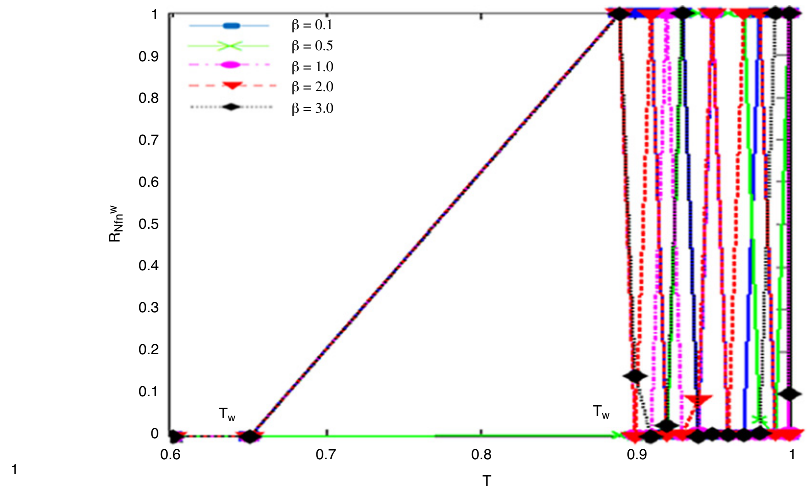

Fig. 29. The results of the shortload cascading failure of the ER networks in which the weights had not be considered, but the shortload had be considered, after the accidental failure.

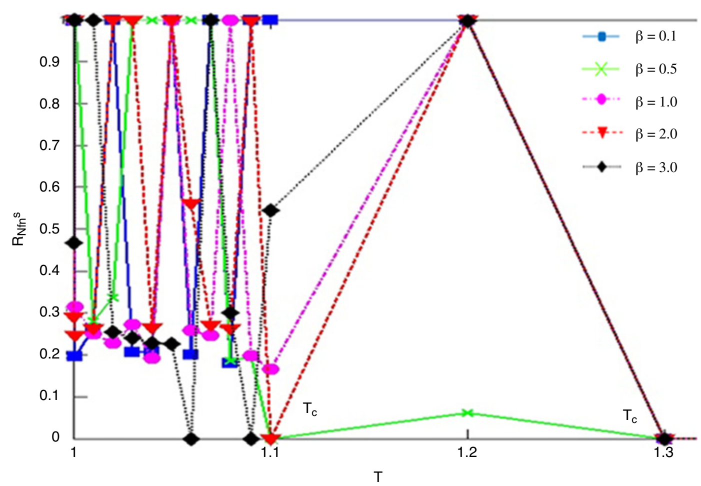

Fig. 30. The results of the accumulative cascading failure of the ER networks in which the weights had not be considered, but the shortload had be considered, after the accidental failure.

failure for the shortload cascading effect. Thus, when $0.9 < T_{shortload} < 1$, for various β and T, the value R^{w}_{Nfn} fluctuated in the [0, 1].

Based on Fig. 29, we can gain the following result. For the ER network in which the weights had not be considered, but the shortload had be considered, when $0.9 < T_{shortload} < 1$, almost all network nodes would possibly not keep the usual function for the shortload after the accidental failure. When $T_{shortload} < 0.9$ (the condition $\beta = 3$ is excepted), almost all the network nodes can be kept from being damaged by the shortload after the accidental failure.

D. The simulation experiment results of the accumulative cascading failure of the ER networks in which the weights had not be considered, but the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R^{s}_{Nfn} (it is shown in Fig. 30).

Obviously, Fig. 30 is also different from Figs. 27 and 28 in the curve characteristic. When $0.9 < T_{shortload} < 1$ and $T_{overload} < 1.3$, the entire network would possibly be paralyzed. This proved our viewpoint that the shortload cascading failure can bring the disastrous results such as the overload.

E. The simulation experiment results of the accumulative cascading failure of the ER networks in which the weights (1–5) and the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R^{s}_{Nfn} (it is shown in Fig. 31).

Obviously, Fig. 31 is also different from Fig. 30. The weights have changed the relations among β, T and R^{s}_{Nfn}. When $0.9 < T_{shortload} < 1$ and $T_{overload} < 1.3$, the entire network would also possibly be paralyzed (but the β value is different

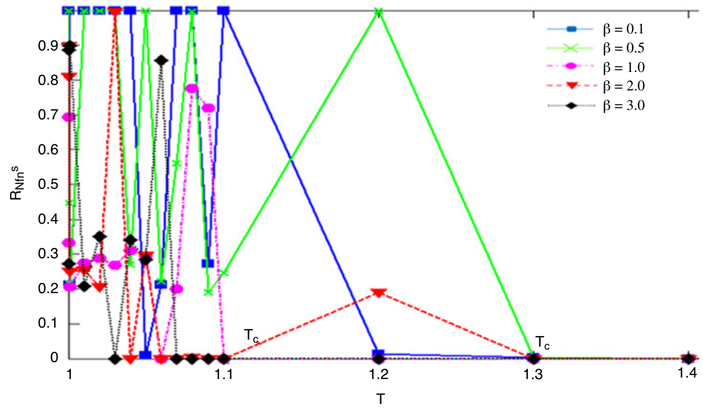

Fig. 31. The results of the accumulative cascading failure of the ER networks in which the weights (1–5) and the shortload had be considered, after the accidental failure.

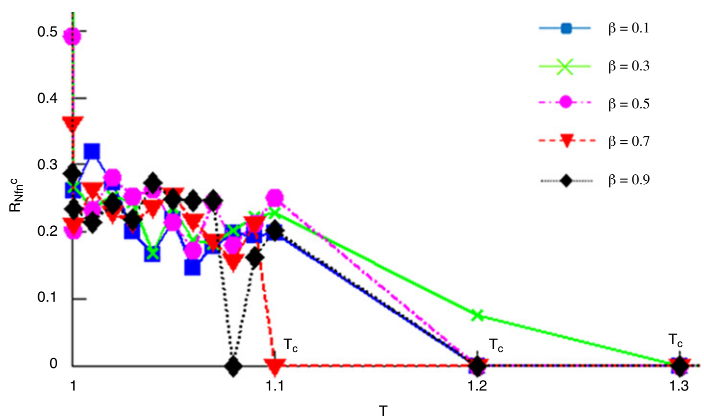

Fig. 32. The results of the overload cascading failure of the BA networks without weights and shortload after the accidental failure.

from the one of Fig. 30). This proved also our idea that the cascading failure law of the weight network is different from the one of the network without the weights.

5.2. The simulation experiment results of the overload and shortload cascading failure of the BA networks

A. The simulation experiment results of the overload cascading failure of the BA networks in which the shortload and the weights need not be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R^{c}_{Nfn} (it is shown in Fig. 32).

Based on Fig. 32, for the BA network in which the shortload and the weights had not be considered, we have discovered that its optimum overload threshold $T_{c2} = 1.1$ can be gained from the control parameter $\beta = 0.7$; otherwise, when $\beta = 0.1$, 0.5 or 0.9, $T_{c2} = 1.2$, even $T_{c2} = 1.3$ (when $\beta = 0.3$). At this overload threshold point, almost all the network nodes can be kept from being damaged by the overload after the accidental failure; otherwise, when $1.01 < T_{overload} < T_{c2}$ (low cost), about 68%–75% of the network nodes can be kept from being damaged.

B. The simulation experiment results of the overload cascading failure of the BA networks in which the weights had not be considered, but the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R^{c}_{Nfn} (it is shown in Fig. 33).

Based on Fig. 33 and Fig. 34, for the BA network in which the weights had not be considered, but the shortload had be considered, we have discovered again that its optimum control parameter $\beta = 0.7$, but at this point its overload threshold T_{c2} is 1.2 and is not, the optimum value. Its the optimum overload threshold T_{c2} is 1.1 (when $\beta = 0.5$), but at this point, its shortload failure size would be very high.

C. The simulation experiment results of the shortload cascading failure of the BA networks in which the weights had not be considered, but the shortload had be considered.

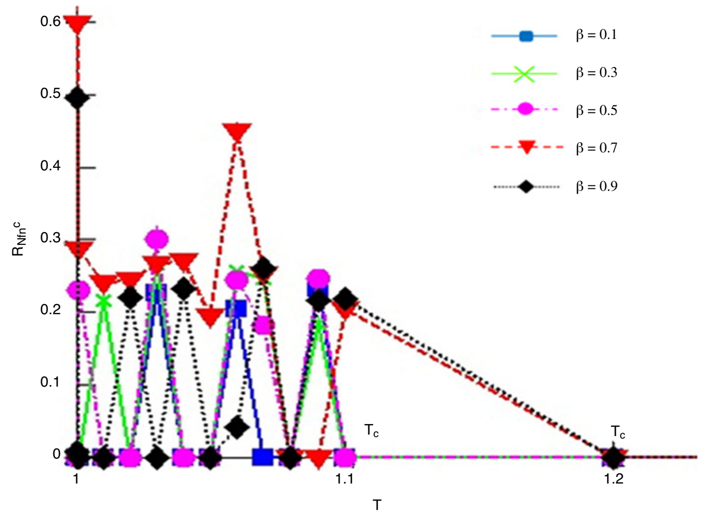

Fig. 33. The results of the overload cascading failure of the BA networks in which the weights had not be considered, but the shortload had be considered, after the accidental failure.

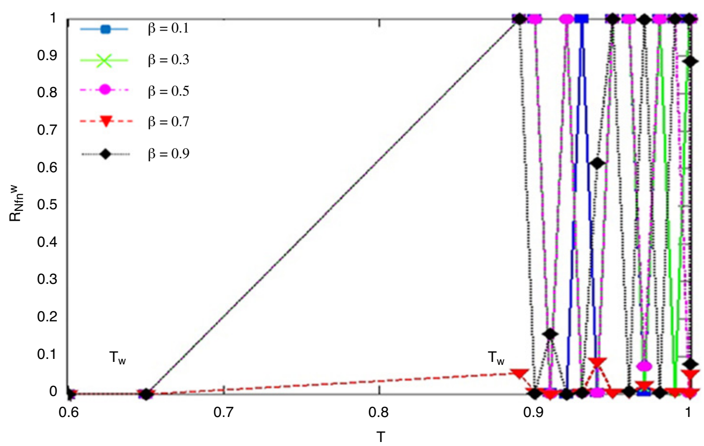

Fig. 34. The results of the shortload cascading failure of the BA networks in which the weights had not be considered, but the shortload had be considered, after the accidental failure.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R_{Nfn}^w (it is shown in Fig. 34).

Obviously, we can easily discover from Fig. 34 that the optimum shortload control parameter β is 0.7 ($\beta = 0.7$) because at this point, even T_w value is very near 1, it can still protect about 90% nodes from being damaged by the shortload after the accidental failure.

D. The simulation experiment results of the accumulative cascading failure of the BA networks in which the weights had not be considered, but the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R_{Nfn}^s (it is shown in Fig. 35).

Obviously, we can still discover easily from Fig. 35 that the optimum overload and shortload control parameter β is 0.7 and its overload threshold $T_{c2} = 1.3$.

E. The simulation experiment results of the accumulative cascading failure of the BA networks in which the weights (1–5) and the shortload had be considered.

In this situation, based on the simulation experiment, we have gained the following curve figure among β, T and R_{Nfn}^s (it is shown in Fig. 36).

Obviously, we can discover again from Fig. 36 that the optimum overload and shortload control parameter β is 0.7, but its overload threshold T_{c2} is only 1.1.

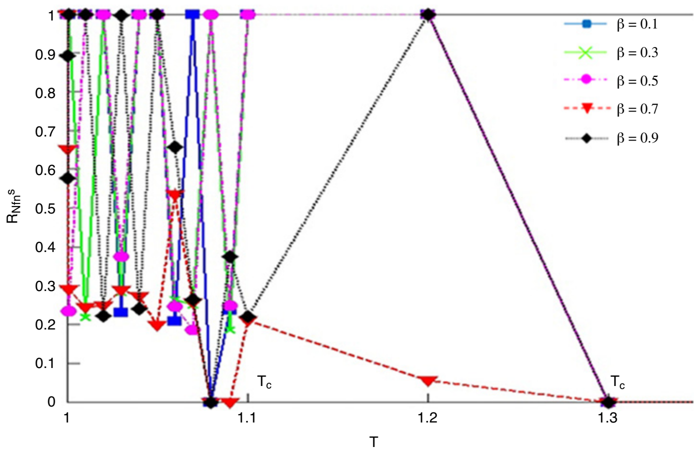

Fig. 35. The results of the accumulative cascading failure of the BA networks in which the weights had not be considered, but the shortload had be considered, after the accidental failure.

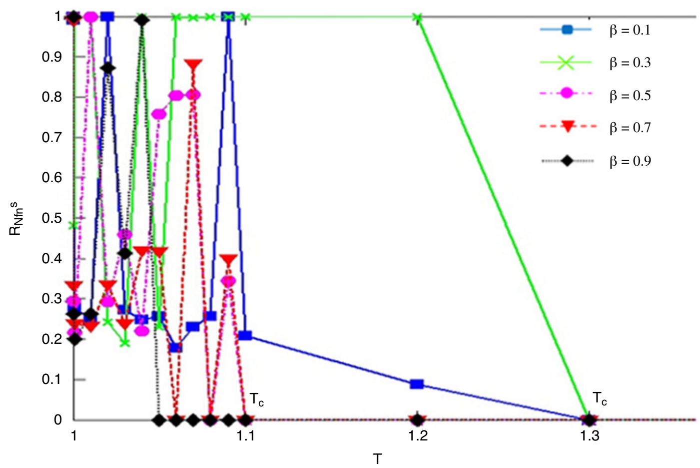

Fig. 36. The results of the accumulative cascading failure of the BA networks in which the weights (1–5) and the shortload had be considered, after the accidental failure.

6. Conclusions and recommendations

In this paper, we first construct the loading-capacity model of the directed and weighted network, then construct the 'over-loading' and 'short-loading' cascading failure models based on the directed and weighted network. Meanwhile, for the two kinds of typical network in real-world – Poisson distribution network and power law distribution network – the paper applied the above models to conduct an in-depth research and numerical analysis. Referencing the results of model and numerical analysis, in order to prevent various kinds of 'over-loading' and 'short-loading' cascading failures in real networks led to large-scale network paralyzed accident, in network construction, we recommend the following.

For various types of power law networks based on 'flow', the power exponent β of 'load-capacity' function should be taken value $(0, 1)$ for a good robustness, and the minimum in-degree and out-degree should be increased respectively, meanwhile, the weight and the scaling exponents of the in-degree and the out-degree distributions should be increased synchronously in the interval $(2, 3)$ as enhancing the resistibility for 'over-loading' and 'short-loading' failures. For the Poisson network, the power exponent β of loading function should be taken value $(0, 3)$ for a good robustness, and the average weight and the average in-degree should be increased respectively restricting $2 < \beta < 3$ as enhancing the resistibility for 'over-loading' and 'short-loading' failures.

Acknowledgments

The authors thank faithfully prof. H. Eugene Stanley for his great guidance, support and help of this paper. Also the authors would like to thank all the reviewers for their constructive comments that helped them improve this manuscript. This work was supported by the State Key Laboratory Open Foundation of Complex Electromagnetic Environment Effects on

Electronics and Information System (CEMEE) of China under Grant No. CEMEE2014K0201A and the National Natural Science Foundation of China under Grant No. 60974080.

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