Chaos, Solitons & Fractals 72 (2015) 4–19

Contents lists available at ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Review

Percolation of interdependent network of networks

Shlomo Havlin [a,*], H. Eugene Stanley [b], Amir Bashan [c], Jianxi Gao [d,e], Dror Y. Kenett [b,*]

[a] Department of Physics, Bar-Ilan University, Ramat Gan, Israel
[b] Center for Polymer Studies, Department of Physics, Boston University, Boston, MA 02215, USA
[c] Channing Division of Network Medicine, Brigham and Women's Hospital, Harvard Medical School, Boston, MA 02115, USA
[d] Department of Automation, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, PR China
[e] Center for Complex Network Research, Department of Physics, Northeastern University, Boston, MA 02115, USA

ARTICLE INFO

Article history:
Available online 18 October 2014

ABSTRACT

Complex networks appear in almost every aspect of science and technology. Previous work in network theory has focused primarily on analyzing single networks that do not interact with other networks, despite the fact that many real-world networks interact with and depend on each other. Very recently an analytical framework for studying the percolation properties of interacting networks has been introduced. Here we review the analytical framework and the results for percolation laws for a *Network Of Networks* (NONs) formed by n interdependent random networks. The percolation properties of a network of networks differ greatly from those of single isolated networks. In particular, because the constituent networks of a NON are connected by node dependencies, a NON is subject to cascading failure. When there is strong interdependent coupling between networks, the percolation transition is discontinuous (first-order) phase transition, unlike the well-known continuous second-order transition in single isolated networks. Moreover, although networks with broader degree distributions, e.g., scale-free networks, are more robust when analyzed as single networks, they become more vulnerable in a NON. We also review the effect of space embedding on network vulnerability. It is shown that for spatially embedded networks any finite fraction of dependency nodes will lead to abrupt transition.

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1. Introduction

The interdisciplinary field of network science has attracted great attention in recent years [1–27]. This has taken place because an enormous amount of data regarding social, economic, engineering, and biological systems has become available over the past two decades as a result of the information and communication revolution brought about by the rapid increase in computing power. The investigation and growing understanding of this extraordinary amount of data will enable us to make the infrastructures we use in everyday life more efficient and more robust. The original model of networks, random graph theory, developed in the 1960s by Erdős and Rényi (ER), is based on the assumption that every pair of nodes is randomly connected with the same probability (leading to a Poisson degree distribution). In parallel, lattice networks in which each node has the same number of links have been used in physics to model physical systems. While graph theory was a well-established tool in the mathematics and computer science literature, it could not adequately describe modern, real-world networks. Indeed, the pioneering observation by Barabási in 1999 [2], that many real networks do not follow the ER model but that organizational principles naturally arise in most systems, led to an overwhelming accumulation of supporting data, new models,

* Corresponding authors.
E-mail addresses: havlins@gmail.com (S. Havlin), drorkenett@gmail.com (D.Y. Kenett).

http://dx.doi.org/10.1016/j.chaos.2014.09.006
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and novel computational and analytical results, and led to the emergence of a new and very active multidisciplinary field: network science.

Significant advances in understanding the structure and function of networks, and mathematical models of networks have been achieved in the past few years. These are now widely used to describe a broad range of complex systems, from techno-social systems to interactions amongst proteins. A large number of new measures and methods have been developed to characterize network properties, including measures of node clustering, node centrality, network modularity, correlation between degrees of neighboring nodes, measures of node importance, and methods for the identification and extraction of community structures. These measures demonstrated that many real networks, and in particular biological networks, contain network motifs—small specific subnetworks—that occur repeatedly and provide information about functionality [9]. Dynamical processes, such as flow and electrical transport in heterogeneous networks, were shown to be significantly more efficient compared to ER networks [28,29].

Complex networks are usually non-homogeneous structures that exhibit a power-law form in their degree (number of links per node) distribution. These systems are called scale-free networks [30]. Some examples of real-world scale-free networks include the Internet [3], the WWW [4], social networks representing the relations between individuals, infrastructure networks such as airlines [31,32], networks in biology, in particular networks of protein–protein interactions [33], gene regulation, and biochemical pathways, and networks in physics, such as polymer networks or the potential energy landscape network. The discovery of scale-free networks has led to a re-evaluation of the basic properties of networks, such as their robustness, which exhibit a character that differs drastically from that of ER networks. For example, while homogeneous ER networks are vulnerable to random failures, heterogeneous scale-free networks are extremely robust [4,5]. Much of our current knowledge of networks is based on ideas borrowed from statistical physics, e.g., percolation theory, fractal analysis, and scaling analysis. An important property of these infrastructures is their stability, and it is thus important that we understand and quantify their robustness in terms of node and link functionality. Percolation theory was introduced to study network stability and to predict the critical percolation threshold [5]. The robustness of a network is usually (i) characterized by the value of the critical threshold analyzed using percolation theory [34] or (ii) defined as the integrated size of the largest connected cluster during the entire attack process [35]. The percolation approach was also extremely useful in addressing other scenarios, such as efficient attacks or immunization [6,8,15,36,37], for obtaining optimal path [38] as well as for designing robust networks [35]. Network concepts were also useful in the analysis and understanding of the spread of epidemics [39,40], and the organizational laws of social interactions, such as friendships [41,42] or scientific collaborations [43]. Moreira et al. investigated topologically-biased failure in scale-free networks and controlled the robustness or fragility by fine-tuning the topological bias during the failure process [44].

Because current methods deal almost exclusively with individual networks treated as isolated systems, many challenges remain [45]. In most real-world systems an individual network is one component within a much larger complex multi-level network (a specific type of a network of networks). As technology has advanced, coupling between networks has become increasingly strong. Node failures in one network will cause the failure of dependent nodes in other networks, and vice versa [46]. This recursive process can lead to a cascade of failures throughout the network of networks system. The study of individual particles has enabled physicists to understand the properties of a gas, but in order to understand and describe a liquid or a solid the interactions between the particles also need to be understood. So also in network theory, the study of isolated single networks brings extremely limited results—real-world noninteracting systems are extremely rare in both classical physics and complex systems. Most real-world network systems continuously interact with other networks, especially since modern technology has accelerated network interdependency.

To adequately model most real-world systems, understanding the interdependence of networks and the effect of this interdependence on the structural and functional behavior of the coupled system is crucial. Introducing coupling between networks is analogous to the introduction of interactions between particles in statistical physics, which allowed physicists to understand the cooperative behavior of such rich phenomena as phase transitions. Surprisingly, preliminary results on mathematical models [46,47] show that analyzing complex systems as a network of coupled networks may alter the basic assumptions that network theory has relied on for single networks. Here we will review the main features of the theoretical framework of Network of Networks, NON [48,49], and present some real world applications.

2. Overview

In order to model interdependent networks, we consider two networks, A and B, in which the functionality of a node in network A is dependent upon the functionality of one or more nodes in network B (see Fig. 1, and vice versa: the functionality of a node in network B is dependent upon the functionality of one or more nodes in network A. The networks can be interconnected in several ways. In the most general case we specify a number of links that arbitrarily connect pairs of nodes across networks A and B. The direction of a link specifies the dependency of the nodes it connects, i.e., link $A_i \rightarrow B_j$ provides a critical resource from node A_i to node B_j. If node A_i stops functioning due to attack or failure, node B_j stops functioning as well but not vice versa. Analogously, link $B_i \rightarrow A_j$ provides a critical resource from node B_i to node A_j.

To study the robustness of interdependent networks systems, we begin by removing a fraction $1 - p$ of network A nodes and all the A-edges connected to these nodes. As an outcome, all the nodes in network B that are dependent

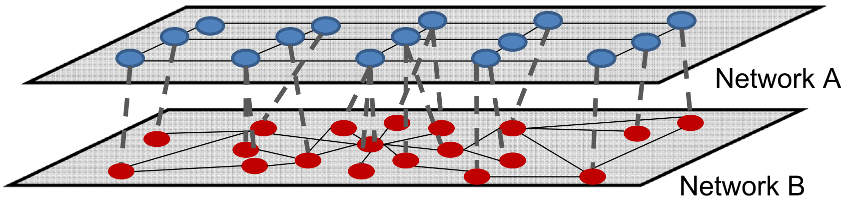

Fig. 1. Example of two interdependent networks. Nodes in network B (communications network) are dependent on nodes in network A (power grid) for power; nodes in network A are dependent on network B for control information.

on the removed A-nodes by $A \rightarrow B$ links are also removed and their B nodes will cause the removal of additional nodes in network A which are dependent on the removed B-nodes by $B \rightarrow A$ links. As a result, a cascade of failures that eliminates virtually all nodes in both networks can occur. As nodes and edges are removed, each network breaks up into connected components (clusters). The clusters in network A (connected by A-edges) and the clusters in network B (connected by B-edges) are different since the networks are each connected differently. If one assumes that small clusters not connected to the giant component become non-functional, this may invoke a recursive process of failures that we now formally describe.

Our insight based on percolation theory is that when the network is fragmented the nodes belonging to the giant component connecting a finite fraction of the network are still functional, but the nodes that are part of the remaining small clusters become non-functional. Thus in interdependent networks only the giant mutually-connected cluster is of interest. Unlike clusters in regular percolation whose size distribution is a power law with a p-dependent cutoff, at the final stage of the cascading failure process just described only a large number of small mutual clusters and one giant mutual cluster are evident. This is the case because the probability that two nodes that are connected by an A-link and their corresponding two nodes are also connected by a B-link scales as $1/N_B$, where N_B is the number of nodes in network B. So the centrality of the giant mutually-connected cluster emerges naturally and the mutual giant component plays a prominent role in the functioning of interdependent networks. When it exists, the networks preserve their functionality, and when it does not exist, the networks split into fragments so small they cannot function on their own.

We ask three questions: What is the critical $p = p_c$ below which the size of any mutual cluster constitutes an infinitesimal fraction of the network, i.e., no mutual giant component can exist? What is the fraction of nodes $P_\infty(p)$ in the mutual giant component at a given p? How do the cascade failures at each step damage the giant functional component?

Note that the problem of interacting networks is complex and may be strongly affected by variants in the model, in particular by how networks and dependency links are characterized. In the following section we describe several of these model variants.

3. Theory of interdependent networks

In order to better understand how present-day crucially-important infrastructures interact, Buldyrev et al. [46] recently developed a mathematical framework to study percolation in a system of two coupled interdependent networks subject to cascading failure. Their analytical framework is based on iterations of the generating function widely used in studies of single-network percolation and single-network structure [43,46,50]. Using the framework to study interdependent networks, we can follow the dynamics of the cascading failures as well as derive analytic solutions for the final steady state. Buldyrev et al. [46] found that interdependent networks were significantly more vulnerable than their noninteracting counterparts. The failure of even a small number of elements within a an interdependent single network in a system may trigger a catastrophic cascade of events that propagates across the global system. For a fully coupled case in which each node in one network depends on a functioning node in other networks and vice versa, Buldyrev et al. [46] found a first-order discontinuous phase transition, which differs significantly from the second-order continuous phase transition found in single isolated networks (Fig. 2. This interesting phenomenon is caused by the presence of two types of links: (i) connectivity links within each network and (ii) dependency links between networks. Parshani et al. [47] showed that, when the dependency coupling between the networks is reduced, at a critical coupling strength the percolation transition becomes second-order.

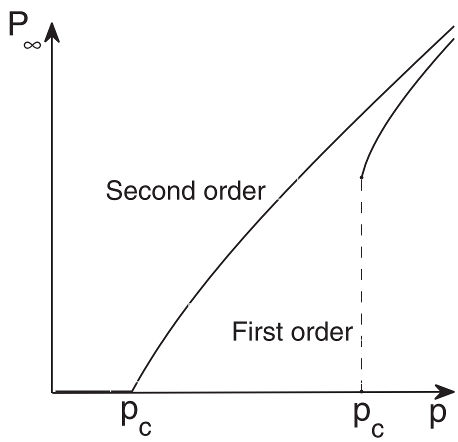

Fig. 2. Schematic demonstration of first and second order percolation transitions. In the second order case, the giant component is continuously approaching zero at the percolation threshold $p = p_c$. In the first order case the giant component approaches zero discontinuously. After Gao et al. [49].

We now present the theoretical methodology used to investigate networks of interdependent networks (see [49]), and provide examples from different classes of networks.

3.1. Generating functions for a single network

We begin by describing the generating function formalism for a single network that is also useful when studying interdependent networks. Here we assume that all N_i nodes in network i are randomly assigned a degree k from a probability distribution $P_i(k)$, and are randomly connected, the only constraint being that the node with degree k has exactly k links [51]. We define the generating function of the degree distribution

$$G_i(x) \equiv \sum_{k=0}^{\infty} P_i(k) x^k, \tag{1}$$

where x is an arbitrary complex variable. The average degree of network i is

$$\langle k \rangle_i = \sum_{k=0}^{\infty} k P_i(k) = \left.\frac{\partial G_i}{\partial x}\right|_{x \to 1} = G_i'(1). \tag{2}$$

In the limit of infinitely large networks $N_i \to \infty$, the random connection process can be modeled as a branching process in which an outgoing link of any node has a probability $kP_i(k)/\langle k \rangle_i$ of being connected to a node with degree k, which in turn has $k-1$ outgoing links. The generating function of this branching process is defined as

$$H_i(x) \equiv \frac{\sum_{k=0}^{\infty} P_i(k) k x^{k-1}}{\langle k \rangle_i} = \frac{G_i'(x)}{G_i'(1)}. \tag{3}$$

The probability f_i that a randomly chosen outgoing link does not lead to an infinitely large giant component satisfies a recursive relation $f_i = H_i(f_i)$. Accordingly, the probability that a randomly chosen node does belong to a giant component is given by $g_i = G_i(f_i)$. Once a fraction $1-p$ of nodes is randomly removed from a network, its generating function remains the same, but must be computed from a new argument $z \equiv px + 1 - p$ [50]. Thus $P_{\infty,i}$, the fraction of nodes that belongs to the giant component, is given by Shao et al. [50],

$$P_{\infty,i} = p g_i(p), \tag{4}$$

where

$$g_i(p) = 1 - G_i[p f_i(p) + 1 - p], \tag{5}$$

and $f_i(p)$ satisfies

$$f_i(p) = H_i[p f_i(p) + 1 - p]. \tag{6}$$

As p decreases, the nontrivial solution $f_i < 1$ of Eq. (6) gradually approaches the trivial solution $f_i = 1$. Accordingly, $P_{\infty,i}$—selected as the order parameter of the transition—gradually approaches zero as in a second-order phase transition and becomes zero when two solutions of Eq. (6) coincide at $p = p_c$. At this point the straight line corresponding to the right hand side of Eq. (6) becomes tangent to the curve corresponding to its left hand side, yielding

$$p_c = 1/H_i'(1). \tag{7}$$

For example, for Erdős–Rényi (ER) networks [52–54], characterized by the Poisson degree distribution,

$$G_i(x) = H_i(x) = \exp[\langle k \rangle_i (x-1)], \tag{8}$$

$$g_i(p) = 1 - f_i(p), \tag{9}$$

$$f_i(p) = \exp\{p \langle k \rangle_i [f_i(p) - 1]\}, \tag{10}$$

and

$$p_c = \frac{1}{\langle k \rangle_i}. \tag{11}$$

Finally, using Eqs. (4), (9), and (10), one obtains a direct equation for $P_{\infty,i}$

$$P_{\infty,i} = p\left[1 - \exp(-\langle k \rangle_i P_{\infty,i})\right]. \tag{12}$$

3.2. Two networks with one-to-one correspondence of interdependent nodes

To initiate and simplify the multitude of problems associated with interacting networks, Buldyrev et al. [46] restricted themselves to the case of two randomly interdependent networks with the same number of nodes, specified by their degree distributions $P_A(k)$ and $P_B(k)$. They also assumed every node in the two networks to have one B → A link and one A → B link connecting the same pair of nodes, i.e., the dependencies between networks A and B establish an isomorphism between them that allows us to assume that nodes in A and B coincide (e.g., are at the same corresponding geographic location—if a node in network A fails, the corresponding node in network B also fails, and vice versa). We also assume, however, that the A-edges and B-edges in the two networks are independent.

Unlike the percolation transition in a single network, the mutual percolation transition in this model is a first-order phase transition at which the order parameter (i.e., the fraction of nodes in the mutual giant component) abruptly drops from a finite value at $p_c + \varepsilon$ to zero at $p_c - \varepsilon$. Here ε is a small number that vanishes as the size of network increases $N \to \infty$. In this range of p, a removal of single critical node may lead to a complete collapse of a seemingly robust network. The size of the largest component drops from NP_∞ to a small value, which rarely exceeds 2. Note that Zhou et al. [55] analyzed this first order transition and found that a simultaneous second order percolation occurs during this abrupt transition.

Note that the value of p_c is significantly larger than in single-network percolation. In two interdependent ER networks, for example, $p_c = 2.4554/\langle k \rangle$, while in a single network, $p_c = 1/\langle k \rangle$. For two interdependent scale-free networks with a power-law degree distribution $P_A(k) \sim k^{-\lambda}$, the mutual percolation threshold is $p_c > 0$, even when $2 < \lambda \leq 3$, when the percolation threshold in a single network is zero.

Note also that, in this new model, networks with a broader degree distribution are less robust against random attack than networks having a narrower degree distribution but the same average degree. This behavior also differs from that found in single networks. To understand this we note that (i) in interdependent networks, nodes are randomly connected—high degree nodes in one network can connect to low degree nodes in other networks, and (ii) at each time step, failing nodes in one network cause their corresponding nodes (and their edges) in the other network to also fail. Thus although hubs in single networks strongly contribute to network robustness, in interdependent networks they are vulnerable to cascading failure. If a network has a fixed average degree, a broader distribution means more nodes with low degree to balance the high degree nodes. Since the low degree nodes are more easily disconnected the advantage of a broad distribution in single networks becomes a disadvantage in interdependent networks.

Buldyrev et al. [46] show that, in a system of two fully interdependent random networks, when the fraction of failed nodes $1-p$ is smaller than a critical value, $p>p_c$, the cascading failures stop after some iterations and a finite fraction of the system, $P_\infty>0$, remains functioning and connected to the giant component. A larger initial damage, $p<p_c$, invokes a cascading failure that fragments the entire system and $P_\infty=0$. Thus, when p approaches p_c from above, the giant component, P_∞, discontinuously jumps to zero in a form of a first order transition. The number of iterations in the cascade, τ, diverges when p approaches p_c, a behavior that was suggested as a clear indication for the transition point in numerical simulations [56].

Among the main features found are the collapse of the system with time in a plateau form (see Fig. 3), and the increase of the plateau time with the system size. Although this phenomena was observed in different models and in real data, its origin remained unclear [46]. To understand the origin of the plateau phenomena, Zhou et al. [55] focused on fully interdependent ER networks. Surprisingly, they find that during the abrupt collapse there appears a hidden spontaneous second order percolation transition that controls the cascading failures, as demonstrated in Fig. 3. It is shown that this simultaneous second order phase transition is the origin of the observed long plateau regime in the cascading failures and its dependence on system size. Moreover, the second order transition sheds light on the critical behavior observed in the collapse of real world systems such as the power law distribution of blackout sizes [57,58,55].

3.3. Framework of two partially interdependent networks

A generalization of the percolation theory for two fully interdependent networks was developed by Parshani et al. [47], who studied a more realistic case of a pair of partially-interdependent networks. Here both interacting networks have a certain fraction of completely autonomous nodes whose function does not directly depend on nodes in the other network. They found that when the fraction of autonomous nodes increases above a certain threshold, the collapse of the interdependent networks characterized by a first-order transition observed in Buldyrev et al. [46] changes, at a critical coupling strength, to a continuous second-order transition as in classical percolation theory [34].

We now describe in more detail the framework developed in Parshani et al. [47]. This framework consists of two networks A and B with the number of nodes $N_{\rm A}$ and $N_{\rm B}$, respectively. Within network A, the nodes are randomly connected by A edges with degree distribution $P_{\rm A}(k)$, and the nodes in network B are randomly connected by B edges with degree distribution $P_{\rm B}(k)$. In addition, a fraction $q_{\rm A}$ of network A nodes depends on the nodes in network B and a fraction $q_{\rm B}$ of network B nodes depends on the nodes in network A. Note that the case of $q_A=q_B=1$ was studied by Buldyrev et al. [46]. We assume that a node from one network depends on no more than one node from the other network, and if A_i depends on B_j, and B_j depends on A_k, then $k=i$. The latter "no-feedback" condition (see Fig. 4 disallows configurations that can collapse without taking into account their internal connectivity [49]. Suppose that the initial removal of nodes from network A is a fraction $1-p$.

We next review the formalism for the cascade process, step by step (see Fig. 5. The remaining fraction of network A nodes after an initial removal of nodes is $\psi'_1\equiv p$. The initial removal of nodes disconnects some nodes from the giant component. The remaining functional part of network A thus contains a fraction $\psi_1=\psi'_1 g_{\rm A}(\psi'_1)$ of the network nodes, where $g_{\rm A}(\psi'_1)$ is defined by Eqs. (5) and (6). Since a fraction $q_{\rm B}$ of nodes from network B depends on

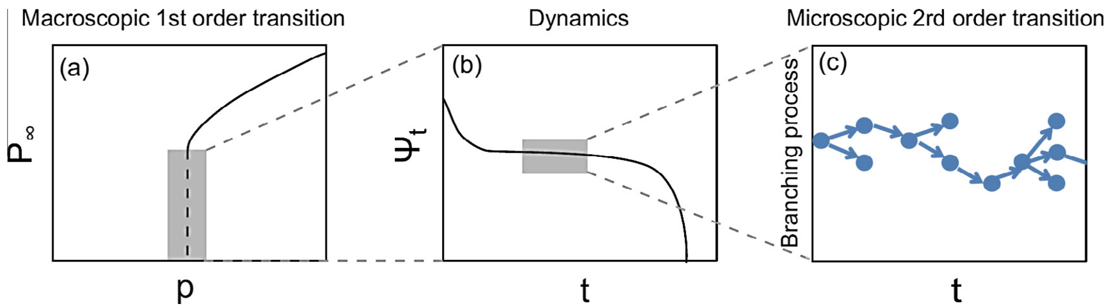

Fig. 3. Demonstration of the simultaneous first and second order transitions in cascading failures of interdependent networks. At the critical point p_c, (a) the mutual giant component has a sudden jump to zero, while (b) the dynamical process of cascading failures is governed by a long plateau stage. In this plateau stage, a second order percolation occurs, which is (c) characterized by a random branching process at criticality, i.e., average branching factor is one. After Zhou et al. [55].

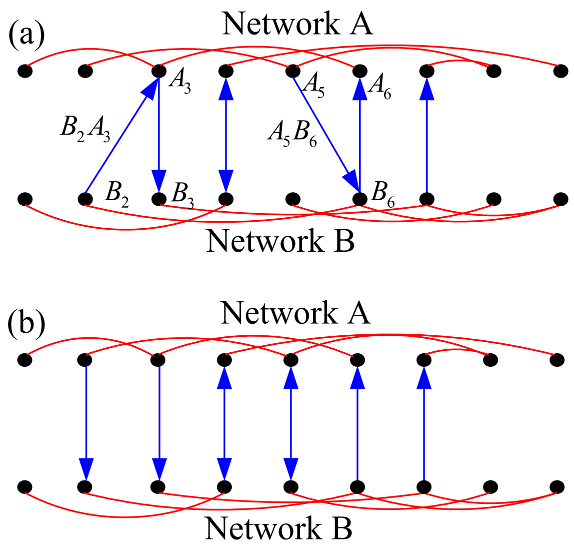

Fig. 4. Description of differences between the (a) feedback condition and (b) no-feedback condition. In the case (a), node A_3 depends on node B_2, and node $B_3 \neq B_2$ depends on node A_3, while in case (b) this is forbidden. In case (a), when $q = 1$ both networks will collapse if one node is removed from one network, which is far from being real. So in our model, we use the no-feedback condition (case (b)). The blue links between two networks show the dependency links and the red links in each network show the connectivity links which enable each network to functional. After Gao et al. [49]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

nodes from network A, the number of nodes in network B that become nonfunctional is $(1-\psi_1)q_B = q_B[1-\psi'_1 g_A(\psi'_1)]$. Accordingly, the remaining fraction of network B nodes is $\phi'_1 = 1 - q_B[1-\psi'_1 g_A(\psi'_1)]$, and the fraction of nodes in the giant component of network B is $\phi_1 = \phi'_1 g_B(\phi'_1)$.

Following this approach we construct the sequence, ψ'_t and ϕ'_t, of the remaining fraction of nodes at each stage of the cascade of failures. The general form is given by

$$\begin{aligned} \psi'_1 &\equiv p, \\ \phi'_1 &= 1 - q_B[1 - p g_A(\psi'_1)], \\ \psi'_t &= p\left[1 - q_A(1 - g_B(\phi'_{t-1}))\right], \\ \phi'_t &= 1 - q_B[1 - p g_A(\psi'_{t-1})]. \end{aligned} \tag{13}$$

To determine the state of the system at the end of the cascade process we look at ψ'_τ and ϕ'_τ at the limit of $\tau \to \infty$. This limit must satisfy the equations $\psi'_\tau = \psi'_{\tau+1}$ and $\phi'_\tau = \phi'_{\tau+1}$ since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at step τ and $\tau + 1$ are equal. Denoting $\psi'_\tau = x$ and $\phi'_\tau = y$, we arrive at the stationary state to a system of two equations with two unknowns [47],

$$\begin{aligned} x &= p\{1 - q_A[1 - g_B(y)]\}, \\ y &= 1 - q_B[1 - g_A(x)p]. \end{aligned} \tag{14}$$

The giant components of networks A and B at the end of the cascade of failures are, respectively, $P_{\infty,A} = \psi_\infty = x g_A(x)$ and $P_{\infty,B} = \phi_\infty = y g_B(y)$. The numerical results were obtained by iterating system (13), where $g_A(\psi'_t)$ and $g_B(\phi'_t)$ are computed using Eqs. (9) and (10). Fig. 6 shows excellent agreement between simulations of cascading failures of two partially interdependent networks with $N = 2 \times 10^5$ nodes and the numerical iterations of system (13). In the simulations, p_c can be determined by the sharp peak in the average number of cascades (iterations), $\langle\tau\rangle$, before the network either stabilizes or collapses [47].

An investigation of Eq. (14) can be illustrated graphically by two curves crossing in the (x, y) plane. For sufficiently large q_A and q_B the curves intersect at two points $(0 < x_0, 0 < y_0)$ and $(x_0 < x_1 < 1, y_0 < y_1 < 1)$. Only the second solution (x_1, y_1) has any physical meaning. As p decreases, the two solutions become closer to each other, remaining inside the unit square $(0 < x < 1; 0 < y < 1)$, and at a certain threshold $p = p_c$ they coincide: $0 < x_0 = x_1 = x_c < 1$, $0 < y_0 = y_1 = y_c < 1$. For sufficiently large q_A and q_B, $P_{\infty,A}$ and $P_{\infty,B}$ as a function of p show a first order phase transition. As q_B decreases, $P_{\infty,A}$ as a function of p shows a second order phase transition. For the graphical representation of Eq. (14) and all possible solutions, see [47].

In a recent study [35,59], it was shown that a pair of interdependent networks can be designed to be more robust by choosing the autonomous nodes to be high degree nodes. This choice mitigates the probability of catastrophic cascading failure.

3.4. Framework for a network of interdependent networks

In many real systems there are more than two interdependent networks, and diverse infrastructures—water and food supply networks, communications networks, fuel networks, financial transaction networks, or power station networks—are coupled together and depend on each other [60]. Understanding the way system robustness is affected by such interdependencies is one of the major challenges when designing resilient infrastructures.

Here we review the generalization of the theory of a pair of interdependent networks [46,61] to a system of n interacting networks [48,62,63], which can be graphically represented (see Fig. 7 as a network of networks (NON). We review an exact analytical approach for percolation of an NON system composed of n *fully* or *partially* coupled randomly interdependent networks. The approach is based on analyzing the dynamical process of the cascading failures. The results generalize the known results for percolation of a single network ($n = 1$) and the $n = 2$ result found in Buldyrev et al. and Parshani et al. [46,47], and show that while for $n = 1$ the percolation transition is a second-order transition, for $n > 1$ cascading failures occur and the transition becomes first-order. Our results for n interdependent networks suggest that the classical percolation theory extensively studied in physics and mathematics is a limiting case of $n = 1$ of a general theory of percolation in NON. As we will discuss here, this general theory has many novel features that are not present in classical percolation theory.

In our generalization, each node in the NON is a network itself and each link represents a *fully* or *partially* dependent pair of networks. We assume that each network i $(i = 1, 2, \ldots, n)$ of the NON consists of N_i nodes linked together by connectivity links. Two networks i and j form a partially dependent pair if a certain fraction $q_{ji} > 0$ of

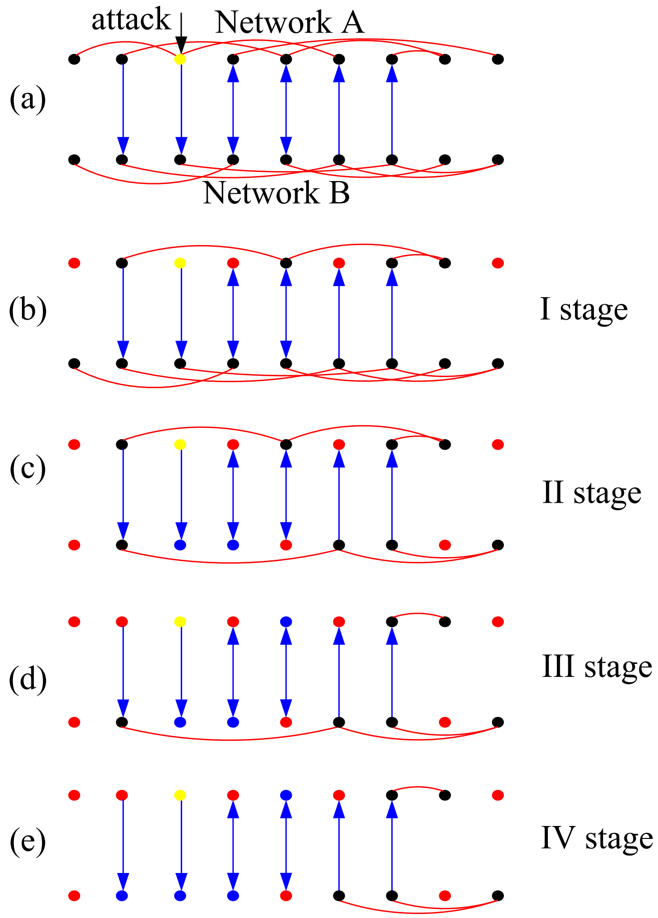

Fig. 5. Description of the dynamic process of cascading failures on two partially interdependent networks, which can be generalized to n partially interdependent networks. In this figure, the black nodes are the survival nodes, the yellow node represents the initially attacked node, the red nodes are the nodes removed because they do not belong to the largest cluster, and the blue nodes are the nodes removed because they depend on the failed nodes in the other network. In each stage, for one network, we first remove the nodes that depend on the failed nodes in the other network or on the initially attacked nodes. Next we remove the nodes which do not belong to the largest cluster of the network. After Gao et al. [49]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

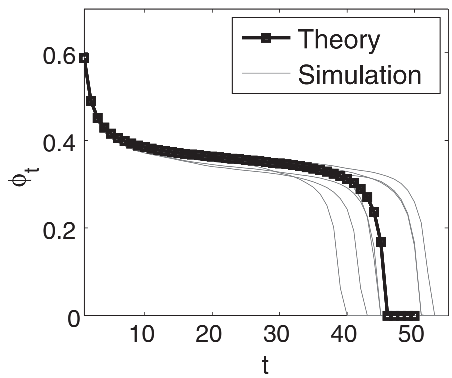

Fig. 6. Cascade of failures in two *partially* interdependent ER networks. The giant component ϕ_t for every iteration of the cascading failures is shown for the case of a first order phase transition with the initial parameters $p = 0.8505$, $a = b = 2.5$, $q_A = 0.7$ and $q_B = 0.8$. In the simulations, $N = 2 \times 10^5$ with over 20 realizations. The gray lines represent different realizations. The squares is the average over all realizations and the black line is the theory, Eq. (13). After Gao et al. [49].

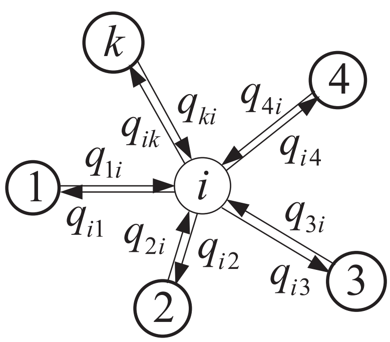

Fig. 7. Schematic representation of a network of networks. Circles represent interdependent networks, and the arrows connect the partially interdependent pairs. For example, a fraction of q_{3i} of nodes in network i depend on the nodes in network 3. The networks which are not connected by the dependency links do not have nodes that directly depend on one another. After Gao et al. [49].

nodes of network i directly depends on nodes of network j, i.e., they cannot function if the nodes in network j on which they depend do not function. Dependent pairs are connected by unidirectional dependency links pointing from network j to network i. This convention indicates that nodes in network i get a crucial commodity from nodes in network j, e.g., electric power if network j is a power grid.

We assume that after an attack or failure only a fraction of nodes p_i in each network i will remain. We also assume that only nodes that belong to a giant connected component of each network i will remain functional. This assumption helps explain the cascade of failures: nodes in network i that do not belong to its giant component fail, causing failures of nodes in other networks that depend on the failing nodes of network i. The failure of these nodes causes the direct failure of dependency nodes in other networks, failures of isolated nodes in them, and further failure of nodes in network i and so on. Our goal is to find the fraction of nodes $P_{\infty,i}$ of each network that remain functional at the end of the cascade of failures as a function of all fractions p_i and all fractions q_{ij}. All networks in the NON are randomly connected networks characterized by a degree distribution of links $P_i(k)$, where k is a degree of a node in network i. We further assume that each node a in network i may depend with probability q_{ji} on only one node b in network j with no feed-back condition.

To study different models of cascading failures, we vary the survival time of the dependent nodes after the failure of the nodes in other networks on which they depend, and the survival time of the disconnected nodes. We conclude that the final state of the networks does not depend on these details but can be described by a system of equations somewhat analogous to the Kirchhoff equations for a resistor network. This system of equations has n unknowns x_i. These represent the fraction of nodes that survive in network i after the nodes that fail in the initial attack are removed and the nodes depending on the failed nodes in other networks at the end of cascading failure are also removed, but without taking into account any further node failure due to the internal connectivity of the network. The final giant component of each network is $P_{\infty,i} = x_i g_i(x_i)$, where $g_i(x_i)$ is the fraction of the remaining nodes of network i that belongs to its giant component given by Eq. (5).

The unknowns x_i satisfy the system of n equations,

$$x_i = p_i \prod_{j=1}^{K} [q_{ji} y_{ji} g_j(x_j) - q_{ji} + 1], \tag{15}$$

where the product is taken over the K networks interlinked with network i by partial dependency links (see Fig. 7) and

$$y_{ij} = \frac{x_i}{p_j q_{ji} y_{ji} g_j(x_j) - q_{ji} + 1}, \tag{16}$$

is the fraction of nodes in network j that survives after the damage from all the networks connected to network j except network i is taken into account. The damage from network i must be excluded due to the no-feedback condition. In the absence of the no-feedback condition, Eq. (15) becomes much simpler since $y_{ji} = x_j$. Eq. (15) is valid for any case of interdependent NON, while Eq. (16) represents the no-feedback condition.

A more general case of interdependency links was studied by Shao et al. [64]. They assumed that a node in network i is connected by s supply links to s nodes in network j from which it gets a crucial commodity. If $s = \infty$, the node does not depend on nodes in network j and can function without receiving any supply from them. The generating function of the degree distribution $P^{ij}(s)$ of the supply links $G^{ji}(x) = \sum_{i=0}^{\infty} P^{ji}(s) x^s$ does not include the term $P^{ji}(\infty) = 1 - q_{ji}$, and hence $G_{ji}(1) = q_{ji} \leqslant 1$. It is also assumed that nodes with $s < \infty$ can function only if they are connected to at least one functional node in network j. In this case, Eq. (15) must be changed to

$$x_i = p_i \prod_{j=1}^{K} \left\{ 1 - G^{ji}[1 - x_j g_j(x_j)\right\}. \tag{17}$$

When all dependent nodes have exactly one supply link, $G_{ij}(x) = x q_{ij}$ and Eq. (18) becomes

$$x_i = p_i \prod_{j=1}^{K} [1 - q_{ji} + q_{ji} x_j g_j(x_j)], \tag{18}$$

analogous to Eq. (15) without the no-feedback condition.

4. Examples of classes of network of networks

We present four examples that can be explicitly solved analytically: (i) a tree-like ER NON *fully* dependent, (ii) a tree-like random regular (RR) NON *fully* dependent, (iii) a loop-like ER NON *partially* dependent, and (iv) an RR network of *partially* dependent ER networks. All cases represent different generalizations of percolation theory for a single network.

4.1. Tree-like NON of ER networks

We solve explicitly the case of a tree-like NON (see Fig. 8) formed by n ER [52–54] networks with average degrees $k_1, k_2, \dots k_i, \dots, k_n$, $p_1 = p$, $p_i = 1$ for $i \neq 1$ and $q_{ij} = 1$ (fully interdependent). Using Eqs. (15) and (16) for x_i and taking into account Eqs. (8)–(10), we find that

$$f_i = \exp\left[-p k_i \prod_{j=1}^{n} (1 - f_j)\right], \quad i = 1, 2, \dots, n. \tag{19}$$

These equations have been solved analytically [48]. They have only a trivial solution ($f_i = 1$) if $p < p_c$, where p_c is the mutual percolation threshold. When the n networks have the same average degree k, $k_i = k$ $(i = 1, 2, \dots, n)$, we obtain from Eq. (19) that $f_c \equiv f_i(p_c)$ satisfies

$$f_c = \exp\left[\frac{f_c - 1}{n f_c}\right]. \tag{20}$$

where the solution can be expressed in terms of the Lambert function $W_-(x), f_c = -\left[n W_-\left(-\frac{1}{n} e^{-\frac{1}{n}}\right)\right]^{-1}$, where $W_-(x)$ is the most negative of the two real roots of the Lambert equation $W(x)\exp[W(x)] = x$ for $x < 0$.

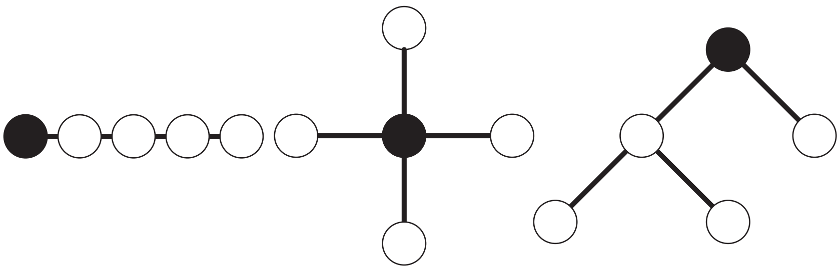

Fig. 8. Three types of loopless networks of networks composed of five coupled networks. All have same percolation threshold and same giant component. The dark node is the origin network on which failures initially occur. After Gao et al. [49].

Once f_c is known, we can obtain p_c and the giant component at p_c:

$$\begin{aligned} p_c &= \left[nkf_c(1-f_c)^{(n-1)}\right]^{-1}, \\ P_\infty(p_c) &= \frac{1-f_c}{nkf_c}. \end{aligned} \tag{21}$$

Eq. (21) generalizes known results for $n = 1, 2$. For $n = 1$, we obtain the known result $p_c = 1/k$, Eq. (11), of an ER network [52–54] and $P_\infty(p_c) = 0$, which corresponds to a continuous second-order phase transition. Substituting $n = 2$ in Eqs. (20) and (21) yields the exact results of Buldyrev et al. [46].

From Eqs. (15) and (16) we obtain an exact expression for the order parameter $P_\infty(p_c)$, the size of the mutual giant component for all p, k, and n values,

$$P_\infty = p[1 - \exp(-kP_\infty)]^n. \tag{22}$$

Solutions of Eq. (22) are shown in Fig. 10(a) for several values of n. Results are in excellent agreement with simulations. The special case $n = 1$ is the known ER second-order percolation law, Eq. (12), for a single network [52–54]. In contrast, for any $n > 1$ the solution of (22) yields a first-order percolation transition, i.e., a discontinuity of P_∞ at p_c.

To analyze p_c as a function of n for different k values, we find f_c from Eq. (20), substitute it into Eq. (21), and obtain p_c. Fig. 10 shows that the NON becomes more vulnerable with increasing n or decreasing k (p_c increases when n increases or k decreases). Furthermore, when n is fixed and k is smaller than a critical number $k_{\min}(n)$, $p_c \geqslant 1$, which means that when $k < k_{\min}(n)$ the NON will collapse even if a single node fails. The minimum average degree $k_{\min}$ as a function of the number of networks is

$$k_{\min}(n) = \left[nf_c(1-f_c)^{(n-1)}\right]^{-1}. \tag{23}$$

Eqs. (19)–(23) are valid for all tree-like structures such as those shown in Fig. 7. Note that Eq. (23) together with Eq. (20) yield the value of $k_{\min}(1) = 1$, reproducing the known ER result, that $\langle k \rangle = 1$ is the minimum average degree needed to have a giant component. For $n = 2$, Eq. (23) also yields results obtained in Buldyrev et al. [46], i.e., $k_{\min} = 2.4554$.

4.2. Tree-like NON of RR networks

We review the case of a tree-like network of interdependent RR networks [48,62] in which the degree of all nodes within each network is assumed to be the same k (Fig. 8). By introducing a new variable $r = f^{\frac{1}{k-1}}$ into Eqs. (15) and (16) and the generating function of RR network [48], the n equations reduce to a single equation

$$r = (r^{k-1} - 1)p(1 - r^k)^{n-1} + 1, \tag{24}$$

which can be solved graphically for any p. The critical case corresponds to the tangential condition leading to critical threshold p_c and P_∞

$$p_c = \frac{r-1}{(r^{k-1}-1)(1-r^k)^{n-1}}, \tag{25}$$

$$P_\infty = p\left(1 - \left\{p^{\frac{1}{n}}P_\infty^{\frac{n-1}{n}}\left[\left(1-\left(\frac{P_\infty}{p}\right)^{\frac{1}{n}}\right)^{\frac{k-1}{k}} - 1\right] + 1\right\}^k\right)^n. \tag{26}$$

Comparing this with the results of a tree-like ER NON, we find that the robustness of n coupled RR networks of degree k is significantly higher than the n interdependent ER networks of average degree k. Although for an ER NON there exists a critical minimum average degree $k = k_{\min}$ that increases with n below which the system collapses, there is no such analogous $k_{\min}$ for a RR NON system. For any $k > 2$, the RR NON is stable, i.e., $p_c < 1$. In general, this is the case for any network with any degree distribution such that $P_i(0) = P_i(1) = 0$, i.e., for a network without disconnected and singly-connected nodes [62].

4.3. Loop-like NON of ER networks

In the case of a loop-like NON (for dependencies in one direction) of n ER networks, all the links are unidirectional and the no-feedback condition is irrelevant (see Fig. 9). If the initial attack on each network is the same $1 - p$, $q_{i-1i} = q_{n1} = q$, and $k_i = k$, using Eqs. (15) and (16) we find that P_∞ satisfies [49]

$$P_\infty = p(1 - e^{-kP_\infty})(qP_\infty - q + 1). \tag{27}$$

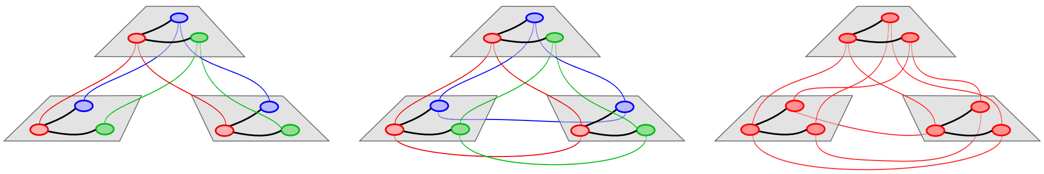

Fig. 9. Illustration of tree-like network of networks and loop-like network of networks. (a) In this tree-like network of networks the mutually interdependent nodes are distinguished by color (red, green and blue) and the tree-like topology guarantees that the size of a mutually interdependent set be exactly n (assuming full interdependency, $q = 1$, as in this example). (b) In this network of networks with loops the dependency behavior is identical to (a) because the added dependency links do not change the partition to sets of mutually interdependent nodes. Thus, with respect to dependency and cascading failures, (b) can be regarded as tree-like. (c) In contrast, if the loops are not closed, a situation can emerge in which all of the nodes are dependent upon one another, i.e., the size of the set of mutually interdependent nodes can be up to $N \times n$. After Shekhtman et al. [65]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Note that when $q = 1$ Eq. (27) has only a trivial solution $P_\infty = 0$, but when $q = 0$ it yields the known giant component of a single network, Eq. (12), as expected. We present numerical solutions of Eq. (27) for two values of q. Note that when $q = 1$ and the structure is tree-like, Eqs. (22) and (26) depend on n, but for loop-like NON structures, Eq. (27) is independent of n.

4.4. NON of ER networks

Now we review results [48,49,63] for a NON in which each ER network is dependent on exactly m other ER networks. This system represents the case of RR network of ER networks. We assume that the initial attack on each network is $1-p$, and each partially dependent pair has the same q in both directions. The n equations of Eq. (15) are exactly the same due to symmetries, and hence P_∞ can be derived analytically,

$$P_\infty = \frac{p}{2^m}\left(1-e^{-kP_\infty}\right)\left[1-q+\sqrt{(1-q)^2+4qP_\infty}\right]^m. \quad (28)$$

Three different behaviors of RR network of ER networks in the different regimes of q can be seen: (i) For $q < q_c$, the percolation is a continuous second order which is characterized by a critical threshold p_c^{II}. (ii) The range of $q_c < q < q_{\max}$ is characterized by an abrupt first order phase transition with a critical threshold p_c^{I}. (iii) For $q > q_{\max}$ no transition exists due to the instant collapse of the system. Furthermore, we obtain the critical threshold for the second order phase transition, p_c^{II} as

$$p_c^{II} = \frac{1}{\bar{k}(1-q)^m}. \quad (29)$$

4.5. NON of RR networks

For a NON composed by n RR networks with the same degree k, where each network depends on exactly m other networks (RR of RR networks), the size of the giant component [63] for all networks follows,

$$1-\left[1-\frac{P_\infty}{p(1-q-qP_\infty)}\right]^{\frac{1}{k}} = p\left\{1-\left[1-\frac{P_\infty}{p(1-q-qP_\infty)}\right]^{\frac{k-1}{k}}\right\}(1-q+qP_\infty)^m. \quad (30)$$

Here again when $m = 0$ or $q = 0$ Eq. (30) reduce to the single network result.

Again, as in the case of the loop-like structure, it is surprising that both the critical threshold and the giant component do not depend on the number of networks n, in contrast to tree-like NON, but only on the coupling q and on both degrees k (intra-links) and m (inter-links). Numerical solutions of Eq. (28) are shown in Fig. 10. In the special case of $m = 0$, Eq. (28) coincide with the known results for a single ER network, Eqs. (11) and (12) separately. It can be shown that when $q < q_c$ we have "weak coupling" represented by a second-order phase transition and when $q_c < q < q_{\max}$ we have "strong coupling" and a first-order phase transition. When $q > q_{\max}$ the system become unstable due to the "very strong coupling" between the networks. In the last case, removal of a single node in one network may lead to the collapse of the NON. These rich generalizations show that the percolation on a single network studied for more than 50 years is a limited case of the more general case of network of networks.

5. Vulnerability of interdependent spatially embedded networks

Current models focus on interdependent networks where space restrictions are not considered. Indeed, in some complex systems the spatial location of the nodes and the actual length of link are not relevant or not even defined, such as in proteins interaction networks [9,66,67] and the World Wide Web [5,68]. However, in many real-world systems, such as power grid networks, ad hoc communication networks and computer networks, nodes and links are located in Euclidian two-dimensional space [69]. Based on universality principles, the dimension of a network is a fundamental quantity to characterize its structure and basic physical properties [34,22]. Indeed, all percolation models whose links have a characteristic length, embedded in space of same dimension belong to *the same* universality class [34]. An example of a spatially embedded network system is power grid networks where the links have a characteristic length since their lengths follow an exponential distribution [22]. Due to universality considerations, any 2d network with links having a characteristics length scale, belong to the same universality class as regular lattices. Thus, to obtain the main features of an arbitrary system of interdependent embedded networks

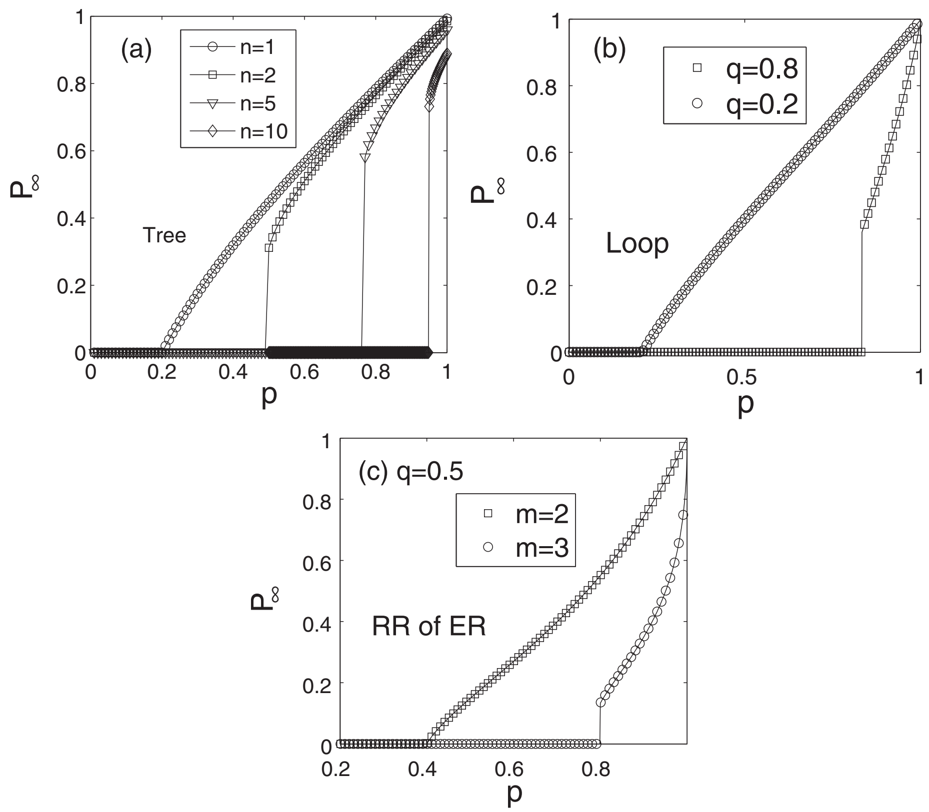

Fig. 10. The fraction of nodes in the giant component P_∞ as a function of p for three different examples of interdependent networks of networks. (a) For a tree-like *fully* ($q = 1$) interdependent NON is shown P_∞ as a function of p for $k = 5$ and several values of n. The results obtained using Eq. (22). Note that increasing n from $n = 2$ yields a first order transition. (b) For a loop-like NON, P_∞ as a function of p for $k = 6$ and two values of q. The results obtained using Eq. (27). Note that increasing q yields a first order transition. (c) For an RR network of ER networks, P_∞ as a function of p, for two different values of m when $q = 0.5$. The results are obtained using Eq. (28), and the number of networks, n, can be any number with the condition that any network in the NON connects exactly to m other networks. Note that changing m from 2 to $m > 2$ changes the transition from second order to first order (for $q = 0.5$). Simulation results are in excellent agreement with theory. After Gao et al. [49].

in two dimensional space, the system has been modeled as two-dimensional lattices. Typically, real spatial networks in two dimensional space are characterized by lower average degree than a square lattice [69]. Thus, the case of coupled lattice is not only a representative example for all its universality class but may serve as a lower bound case, while real coupled spatial networks are even more vulnerable.

Here, we review recent analytical and numerical results recently presented by Bashan et al. on the stability of systems of two interdependent spatially embedded networks, modeled as two interdependent lattices [70]. It is found that in such systems $q_c = 0$, i.e., any coupling $q > 0$ leads to an abrupt first-order transition. It is shown that the origin for this extreme vulnerability of spatially embedded networks lies in the critical behavior of percolation of a single lattice, which is characterized by order critical exponent $\beta < 1$ [34,71]. This is in contrast to random networks for which $\beta = 1$, leading to $q_c > 0$ for interdependent random networks. Here the dependency links are between lattices' nodes located in different random spatial positions (Fig. 11(a)) or between lattice nodes and nodes of random networks where the space does not play a role at all (Fig. 11(b)). In the case of dependency links between lattice nodes with exactly the same position, the transition is always continuous, as for percolation in a single lattice [72]. Note that the fully interdependent limit of $q = 1$ of coupled lattices was studied by Li et al. [73].

The theoretical and numerical approaches predict [70] that a real-world system of interdependent spatially embedded networks which are characterized by $\beta < 1$ will, for any $q > 0$, abruptly disintegrate. Since for percolation of lattice networks it is known that for any dimension $d < 6,\ \beta < 1$ [34], we expect that also interdependent systems embedded in $d = 3$ (or any $d < 6$) will collapse abruptly for any finite fraction of dependency q. Indeed, Dobson et al. [74] analyze the statistics of many real world outages events and show that they are commonly resulted by cascading failure. Our results show that an important possible mechanism in these events is the interdependencies between nodes in spatial networks.

Consider a system of two interdependent networks, $i = 1$ and $i = 2$, where a fraction $1 - p_i$ of nodes of each network is initially randomly removed. We assume that only the nodes which belong to the giant component of the remaining networks which constitute a fraction $P_{\infty,i}(p_i)$ of the original network remain functional. Each node that has been removed or disconnected from the

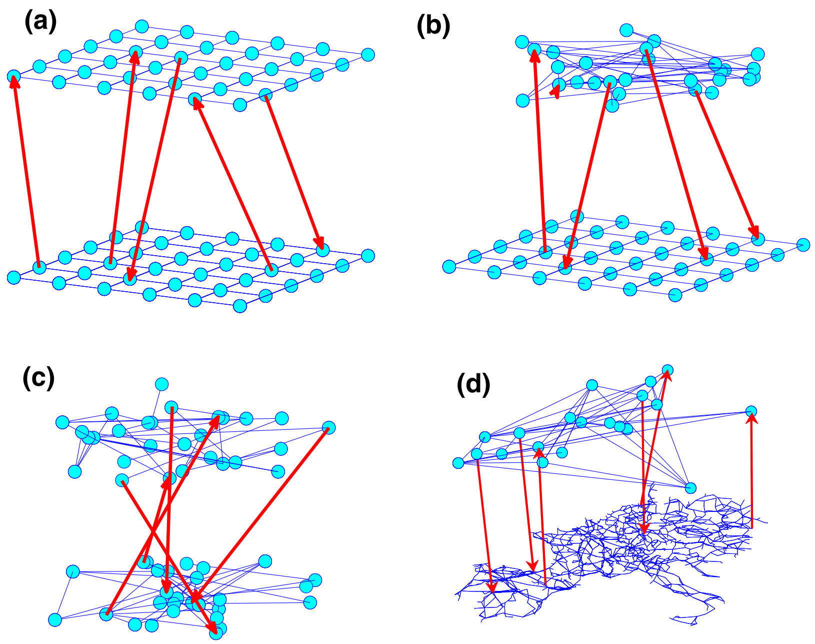

Fig. 11. A system of interdependent networks is characterized by the structure (dimension) of the single networks as well as by the coupling between the networks. In random networks with no space restrictions, such as ER and RR, the connectivity links (blue lines) do not have a defined length. In contrast, in spatially embedded networks nodes are connected only to nodes in their geometrical neighborhood creating a two-dimensional network, modeled here as a square lattice. The (red) arrows represent directed dependency relations between nodes in different networks, which can be of different types: (a) coupled lattices (b) coupled lattice-random network (c) coupled random networks (d) real-world spatial network (European power grid) coupled with random network. Models (b) and (d) belong to the same universality class. After Bashan et al. [70]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

giant component causes its dependent node in the other network to also fail. This leads to further disconnections in the other network and to cascading failures. The size of the networks' giant components at the end of the cascade is given by $P_{\infty,i}(x_i)$, where x_i are the solutions of the self consistent equations [48]

$$x_1 = p_1 q_1 P_{\infty,2}(x_2) + p_1(1 - q_1), \tag{31}$$

$$x_2 = p_2 q_2 P_{\infty,1}(x_1) + p_2(1 - q_2), \tag{32}$$

where q_i is the fraction of nodes in network i which depends on nodes in the other network. Here we assume no restrictions on the selection of the directed dependency links. The results for the case of "no-feedback-condition", where the dependency links are bidirectional [48], are qualitatively the same. The function $P_{\infty,i}(x)$ can be obtained either analytically or numerically from the percolation behavior of a *single* network.

For simplicity, we focus on a symmetric case, where both networks have the same degree distribution $P(k)$ and same topology, and where $p_1 = p_2 \equiv p$ and $q_1 = q_2 \equiv q$. Still, the results are valid for any system of interdependent spatially embedded networks (like planar graph) which belong to the same universality class. In particular, in order to study the role of spatial embedding, the percolation transition in the case of a pair of interdependent lattices is compared (Fig. 11(a)) to the case of a pair of interdependent random-regular (RR) networks (Fig. 11(c)). The RR networks have the same degree distribution, $P(k) = \delta_{k,4}$, as for the lattices with the only difference that the lattice-networks are embedded in space, in contrast to RR networks.

In the symmetric case, Eqs. (31) and (32) can be reduced to a single equation

$$x = pqP_\infty(x) + p(1 - q), \tag{33}$$

where the size of the giant component at steady state is $P_\infty(x)$. For any values of p and q, the solution of Eq. (33) can be graphically presented as the intersection between the curve $y = pqP_\infty(x) + p(1 - q)$ and the straight line $y = x$ representing the right-hand-side and the left-hand-side of Eq. (33) respectively. The form of $P_\infty(x)$ for conventional percolation is obtained from numerical simulations of a single lattice and analytically for a single RR network [75,76]. From the solution of Eq. (33) we obtain $P_\infty(p)$ as a function of p for several values of q. This $P_\infty(p)$ is the new percolation behavior for a system of interdependent networks, shown in Fig. 13(a), for the case of coupled lattices and in Fig. 13(b) for the case of coupled RR networks. In the case of interdependent lattices, only for $q = 0$, no coupling between the networks (the single network limit), the transition is the conventional second-order percolation transition, while for any $q > 0$ the collapse is abrupt in the form of first-order transition. In marked contrast, in the case of interdependent RR networks, for $q > q_c \cong 0.43$ the transition is abrupt, while for $q < q_c$ the transition is continuous.

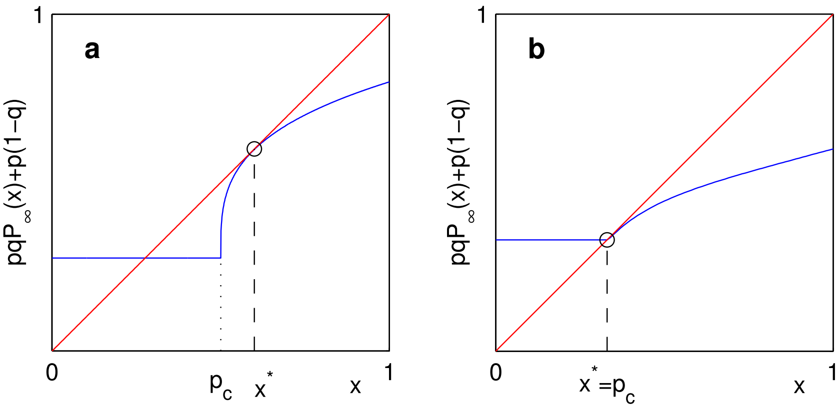

Fig. 12. Schematic solution of the critical point of (a) coupled lattices and (b) coupled random-regular (RR) networks. The left-hand-side and right-hand-side of Eq. (33) are plotted as a straight (red) line and a (blue) curve respectively. The tangential touching point, x^*, marked with a (black) circle, represents the new percolation threshold in the system of interdependent networks. In the case of coupled lattices (panel a), due to the infinite slope of the curve at p_c, x^* is always larger than p_c and, thus, there is always (for any $q > 0$) a discontinuous jump in the size of the giant component as p decreases. In contrast, in coupled random networks (panel b) the slope of the curves is finite for any value of x. Therefore, there exist $q < q_c$ for which x^* is equal to p_c, leading to a continuous behavior in the network's size. After Bashan et al. [70]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

A discontinuity of $P_\infty(p)$ is a result of a discontinuity of $x(p)$, represented graphically as the tangential touching point of the curve and the straight line (see schematic representation in Fig. 12). At this point, $p \equiv p^\star$ is the new percolation threshold in the case of interdependent networks, and $x = x^\star$ yields the size of the giant component at the transition, $P_\infty^\star \equiv P_\infty(x^\star)$, which abruptly jumps to zero as p slightly decreases. The condition for a first-order transition at $p = p^\star$, for a given q, is thus given by solving Eq. (33) together with its tangential condition,

$$1 = p^\star q P'_\infty(x^\star). \tag{34}$$

The size of the giant component at the transition $P_\infty^\star$ depends on the coupling strength q such that reducing q leads to smaller value of $x^\star$ and thus smaller discontinuity in the size of the giant component. In general, $P_\infty(x)$ of a single network has a critical threshold at $x = p_c$ such that $P_\infty(x \leqslant p_c) = 0$ while $P_\infty(x > p_c) > 0$ and monotonically increases with x [34]. As long as $x^\star > p_c$, the size of the discontinuity is larger then zero. However, for a certain

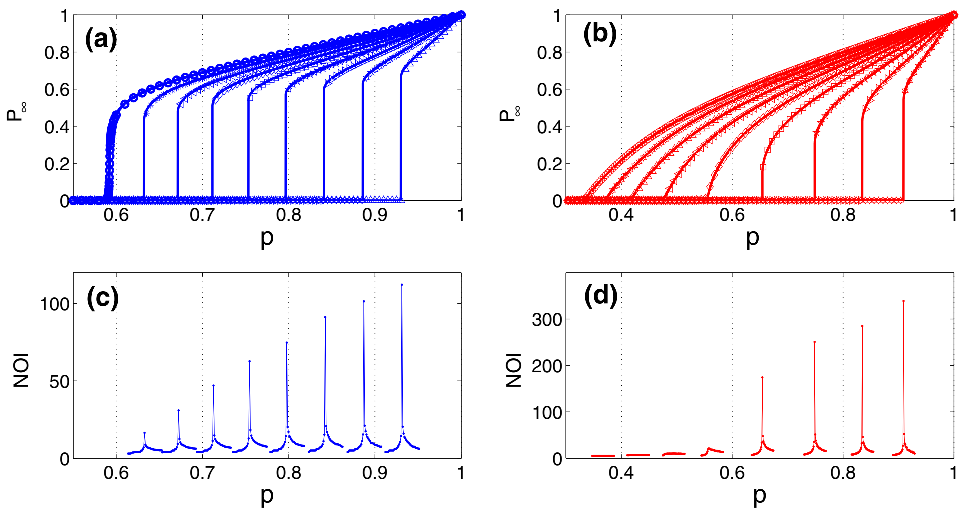

Fig. 13. Percolation transition of interdependent lattices compared to interdependent random networks. The size of the giant component P_∞ at steady state after random failure of a fraction $1 - p$ of the nodes of (a) two interdependent lattice networks with periodic boundary conditions (PBC) and (b) two random-regular (RR) networks. All networks are of size 16×10^6 nodes and the same degree distribution $P(k) = \delta_{k,4}$. The coupling between the lattices and between the RR networks changes from $q = 0$ to $q = 0.8$ with step 0.1 (from left to right). The solid lines are the solutions of Eq. (33) and the symbols represent simulation results. In the case of interdependent lattices, only for $q = 0$ (no coupling, i.e., a single lattice) the transition is the conventional second-order percolation, while for any $q > 0$ the collapse is abrupt in the form of first-order transition. This is in marked contrast to the case of interdependent RR networks, where only for $q > q_c \cong 0.43$ the transition is abrupt, while for $q < q_c$ the transition is continuous. A characteristic behavior in a first-order percolation transition in coupled networks is the sharp divergence of the number of iterations (NOI) when p approaches $p_c^\star$ [56] as seen for (c) coupled lattices for any $q > 0$ and for (d) coupled RR networks for $q > q_c$. Models of coupled lattices with PBC have the same behavior as models without. After Bashan et al. [70].

critical coupling $q \equiv q_c$, $x^\star \to p_c$ and the size of the jump becomes zero. In this case the percolation transition becomes continuous.

Therefore, the critical dependency q_c below which the discontinuous transition becomes continuous, must satisfy Eqs. (33) and (34) for $x \to p_c$ given by

$$p_c = p_c^\star(1 - q_c), \tag{35}$$

$$1 = p_c^\star q_c P'_\infty(p_c). \tag{36}$$

A dramatic different behavior between random and spatial coupled networks is derived from Eq. (36). This difference is a consequence of the critical behavior of percolation in a single network. In the case of a single random network $P'_\infty(x)$ is finite for any value of x. This allows an exact solution of Eq. (36), yielding a finite non-zero value for q_c. However, for the case of a single lattice network the derivative of $P_\infty(x)$ diverges at the critical point, $P'_\infty(p_c) = \infty$, yielding $q_c = 0$. Therefore, from Eq. (36) follows that any coupling $q > 0$ between lattices leads to an abrupt first order transition, as indeed suggested by simulations reported in Fig. 13.

The behavior of the percolation order parameter of a single network near the critical point is defined by the critical exponent β, where $P_\infty(x \to p_c) = A(x - p_c)^\beta$. Since for single 2d lattice $\beta = 5/36 < 1$, it follows that $P'_\infty(x)$ diverges for $x \to p_c$ for all networks embedded in two dimensional space [34,71]. In contrast, for random networks, such as Erdös–Rényi (ER) and Random-Regular (RR), $\beta = 1$ which yields a finite value of $P'_\infty(p_c)$ [34,71] and therefore a finite value for q_c. The coupled embedded networks case has been studied by Bashan et al. [70], and was generalized by Shekhtman et al. [65] to percolation of network of networks.

6. Summary

In summary, this paper presents a review of the recently-introduced mathematical framework of for percolation of a Network of Networks (NON). In interacting networks, when a node in one network fails it usually causes dependent nodes in other networks to fail which, in turn, may cause further damage in the first network and result in a cascade of failures with catastrophic consequences. Our analytical framework enables us to follow the dynamic process of the cascading failures step-by-step and to derive steady state solutions. Interdependent networks appear in all aspects of life, nature, and technology. Examples include (i) transportation systems such as railway networks, airline networks, and other transportation systems [61,77]; (ii) the human body as studied by physiology, including such examples of interdependent NON systems as the cardiovascular system, the respiratory system, the brain neuron system, and the nervous system [78]; (iii) protein function as studied in biology, treating protein interaction—the many proteins involved in numerous functions—as a system of interacting networks; (iv) the interdependent networks of banks, insurance companies, and business firms as studied by economics; (v) species interactions and the robustness of interaction networks to species loss as studied in ecology, in which is essential to understand the effects of species decline and extinction [79]; and (vi) the topology of statistical relationships between distinct climatologically variables across the world as studied by climatology [80].

Thus far only a few real-world interdependent systems have been thoroughly analyzed [61,77]. We expect our work to provide insights leading further analysis of real data on interdependent networks. The benchmark models presented here can be used to study the structural, functional, and robustness properties of interdependent networks. Because in real-world NONs individual networks are not randomly connected and their interdependent nodes are not selected at random, it is crucial that we understand the many types of correlation that exist in real-world systems and that we further develop the theoretical tools to take them into account. Further studies of interdependent networks should focus on (i) an analysis of real data from many different interdependent systems and (ii) the development of mathematical tools for studying real-world interdependent systems. Many real networks are embedded in space, and the spatial constraints strongly affect their properties [22,69,73]. There is a need to understand how these spatial constraints influence the robustness properties of interdependent networks [77]. Other properties that influence the robustness of single networks, such as the dynamic nature of the configuration in which links or nodes appear and disappear and the directed nature of some links, as well as problems associated with degree–degree correlations and clustering, should be also addressed in future studies of coupled network systems. Additional critical issues are the improvement of the robustness of interdependent infrastructures, self healing approaches and efficient mitigation of cascading failures. The reviewed studies thus far have shown that there are three methods of achieving the goal of improving robustness (i) by increasing the fraction of autonomous nodes [47], (ii) by designing dependency links such that they connect the nodes with similar degrees [81,61], and (iii) by protecting the high-degree nodes against attack [35]. Achieving the above mentioned goals will provide greater safety and stability in today's socio-techno world.

Networks dominate every aspect of present-day living. The world has become a global village that is steadily shrinking as the ways that human beings interact and connect multiply. Understanding these connections in terms of interdependent networks of networks will enable us to better design, organize, and maintain the future of our socio-techno-economic world.

Acknowledgement

SH, HES, and DYK thank DTRA, ONR, BSF, the LINC (No. 289447) and the Multiplex (No. 317532) EU projects, the DFG, and the Israel Science Foundation for support. JG thank the support from National Natural Science Foundation of China (Grant No. 61374160).

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