General-R High-Temperature Series for the Susceptibility, Second Moment, and Specific Heat of sc and fcc Ising Models with Lattice Anisotropy

Fredric Harbus and H. Eugene Stanley
Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 30 August 1972)

Of particular current interest is the critical behavior of functions on crossing over from one lattice dimensionality to another. To this end, we report high-temperature series for an Ising model with lattice anisotropy—a.i., with different exchange constants for different lattice directions. The Hamiltonian is

$$\mathcal{H}_{\text{anis}} = -J_{xy} \sum_{(i)} s_i s_j - J_\perp \sum_{(i)} s_i s_{i+1} - J_x \sum_{(i)} s_i s_{i+1} + R \sum_{(i)} s_i s_{i+1}$$

where \( s_i = \pm 1 \), the first sum is over all nearest-neighbor pairs in the \( xy \) plane, and the second sum is over pairs coupled in the \( z \) direction. The susceptibility, second moment, and specific-heat series are explicitly presented for arbitrary \( J_{xy} \) and \( J_x \) for the simple cubic (sc) and face-centered cubic (fcc) lattices to tenth order in inverse temperature. The general-\( R \) series are essential if one wishes to study the Riedel–Wegner crossover exponent appropriate to changing lattice dimensionality, since for \( R = J_x/J_{xy} = 0 \), both the sc and fcc lattices reduce to two-dimensional square lattices, while in the limit \( R \to \infty \), the sc reduces to noninteracting linear chains.

I. INTRODUCTION

Ising-model Hamiltonians with “lattice anisotropy,” i.e., different exchange constants in different lattice directions, have recently been considered in two different but related contexts in the field of critical phenomena. The first context concerns testing the “universality hypothesis,” which was put forth to describe just which features of the interaction Hamiltonian determine the critical indices. For example, according to universality, the exponents should retain their values for the nearest-neighbor (nn) isotropic model Hamiltonian when second-neighbor interactions or unequal exchange constants on the lattice are introduced. A change in the exponents is expected, however, when the effective dimensionality of the system is altered.

In their investigations of these predictions, various authors have utilized high-temperature series expansions to study the Ising Hamiltonian

$$\mathcal{H} = -J_{xy} \sum_{(i)} s_i s_j - J_x \sum_{(i)} s_i s_{i+1}$$

where \( s_i = \pm 1 \), the first sum is over all nn pairs in an \( x-y \) plane, and the second sum is over all nn pairs whose relative displacement vector has a \( z \) component. High-temperature series were analyzed for a range of values of the parameter \( R \) by both groups. In one case, conclusions consistent with universality are reached, i.e., for all \( R > 0 \) the indices are three dimensional, and at \( R = 0 \) they change discontinuously to their two-dimensional values. The other work claims to find exponents varying continuously with \( R \) for small \( R \), in violation of the universality hypothesis. In view of this discrepancy in the literature (between Refs. 4 and 5), other workers should have available to

| Table I. Coefficients \( a_{nj} \) in reduced susceptibility series for the sc lattice, 

\[ \chi = k_B T \chi / N \mu^2 = \sum_{n,j} a_{nj} \tanh^{-1}(\beta J_{xy}) \tanh^{-1}(\beta J_x). \] 

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>16</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>80</td>
<td>32</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>336</td>
<td>240</td>
<td>48</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>276</td>
<td>1264</td>
<td>1392</td>
<td>512</td>
<td>64</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>740</td>
<td>4432</td>
<td>6688</td>
<td>8888</td>
<td>8888</td>
<td>80</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1772</td>
<td>14768</td>
<td>29136</td>
<td>23690</td>
<td>8544</td>
<td>1376</td>
<td>96</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>5172</td>
<td>47376</td>
<td>11528</td>
<td>124720</td>
<td>63216</td>
<td>16080</td>
<td>1968</td>
<td>112</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>13492</td>
<td>147504</td>
<td>442368</td>
<td>593856</td>
<td>400022</td>
<td>142416</td>
<td>27216</td>
<td>2672</td>
<td>128</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>34876</td>
<td>448336</td>
<td>1595896</td>
<td>2621232</td>
<td>2224312</td>
<td>1054864</td>
<td>281008</td>
<td>42672</td>
<td>3480</td>
<td>144</td>
<td>2</td>
</tr>
</tbody>
</table>

365
them series for general R in order to conduct further tests of universality. While Ref. 4 obtained such series only for the simple cubic (sc) susceptibility $\chi^{(3)}$, we present here five additional general-R series: $\chi^{(2)}$, $\mu_2$, $\mu_2^R$, $C_R$, $C_H$, and $C_{HR}$, where $\chi$, $\mu_2$, and $C_H$ (the reduced susceptibility, second moment of the correlation function, and the reduced specific heat) are defined below in Eqs. (2.1)-(2.3), respectively.

The anisotropic Hamiltonian (1.1) has also received considerable attention in connection with a “crossover” exponent $\alpha_R^{a_{i}T}$ $\varphi$ and, in our view, an even more compelling motivation to focus upon the general-R series is their applicability to calculations of $\varphi$. The exponent $\varphi$ describes the singular behavior of the quantity $T_{c}(R) - T_{c}(R = 0)$ as the system “crosses over” from one universality class to another, i.e., as the system approaches its two-dimensional limit ($R = 0$).

To derive the equation defining $\varphi$, we start from the assumption that the Gibbs potential is a generalized homogeneous function (GHF) in the variable $R$ as well as in the variables $\tau = [T - T_{c}(0)]/T_{c}(0)$ and $H$,

$$G(\lambda^{a} \varphi, \lambda^{a} H, \lambda^{a} R) = \lambda G(\tau, H, R).$$  \hspace{1cm} (1.2)

Equation (1.2) is assumed to hold for all $\lambda > 0$ and small values of the arguments (i.e., in the vicinity of the two-dimensional critical point $\tau = H = R = 0$).

Now, for a given value of $R_o > 0$, the right-hand side of (1.2) certainly has a singularity at the value of temperature critical for a system with this $R$, that is, for a value $\tau_o(R_o) = [T_{c}(R_o) - T_{c}(0)]/T_{c}(0) = c R_o$, where $c$ is some constant. But then by the functional form of (1.2), there exists an entire line of singularities given by $\tau_o(R) \sim R^{a_{i}^{R}/a_{i}^{R}}$, or

$$T_{c}(R) - T_{c}(0) \sim R^{1/\varphi},$$  \hspace{1cm} (1.3)

where $\varphi$ is defined to be $a_{R}/a_{T}$.

A second prediction of the GHF hypothesis for the parameter $R$ is that there exists a constant “gap” exponent for successive derivatives with respect to $R$ of the thermodynamic functions derived from the Gibbs potential. Consider, e.g., the reduced susceptibility $\chi$, for which we define

$$\chi^{(0)} = \left( \frac{\partial^{a} \chi}{\partial R^{a}} \right)_{T, H},$$  \hspace{1cm} (1.4)

First we find the two-dimensional susceptibility exponent $\gamma_2 = \frac{1}{2}$ in terms of the scaling powers. To do this, differentiate (1.2) twice with respect to $H$ and set $H = 0$,

$$\chi^{(0)}(\lambda^{a} \varphi, 0, \lambda^{a} R) = \lambda^{2 a_{R}^{2}} \chi^{(0)}(\tau, 0, R).$$  \hspace{1cm} (1.5)

Now setting $\lambda^{a} \varphi$ equal to a small positive constant, and letting $R = 0$, we get the asymptotic form of $\chi^{(0)}$
TABLE III. Coefficients \( b_n \) in second-moment series for the sc lattice,

\[
\mu_2 = \frac{1}{\bar{\chi}} \frac{1}{\bar{C}_2(\bar{T})} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} b_{nj} \tanh^{n+1}(gJ_{xy}) \tanh^{j}(gJ_x),
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( j )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>32</td>
<td>32</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>164</td>
<td>272</td>
<td>128</td>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>704</td>
<td>1696</td>
<td>1248</td>
<td>352</td>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2708</td>
<td>8816</td>
<td>9168</td>
<td>4032</td>
<td>768</td>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>9696</td>
<td>40608</td>
<td>56160</td>
<td>34656</td>
<td>10368</td>
<td>1440</td>
<td>72</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>32948</td>
<td>171504</td>
<td>301488</td>
<td>245936</td>
<td>103328</td>
<td>22688</td>
<td>2432</td>
<td>98</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>107648</td>
<td>678432</td>
<td>1468704</td>
<td>1517408</td>
<td>840576</td>
<td>259744</td>
<td>44128</td>
<td>3808</td>
<td>128</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>346916</td>
<td>2540552</td>
<td>6630368</td>
<td>6412544</td>
<td>5889248</td>
<td>2398416</td>
<td>374320</td>
<td>78512</td>
<td>5632</td>
<td>162</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1023960</td>
<td>9193120</td>
<td>28179532</td>
<td>42859360</td>
<td>56779200</td>
<td>18910752</td>
<td>3596816</td>
<td>1149792</td>
<td>130356</td>
<td>7968</td>
<td>200</td>
</tr>
</tbody>
</table>

\[
\bar{x}^{(0)} - \tau^{(1-2s_R)/s_R} = \tau^{-\gamma_0}.
\]

Differentiating Eq. (1.5) \( n \) times with respect to \( R \) changes the scaling power on the right-hand side by \( na_R \), so that

\[
\bar{x}^{(0)}(\tau, 0, \lambda^{s_R}R) = \lambda^{-1+2s_R-2s_R} \bar{x}^{(0)}(\tau, 0, R),
\]

Then setting \( R = 0 \),

\[
\bar{x}^{(0)}(\tau, 0, 0) \sim \tau^{-(1-2s_R-2s_R)/s_R} = \tau^{-\gamma_0},
\]

where

\[
\gamma_0 = \gamma_0 + n\varphi.
\]

Similar equations hold for the specific-heat derivatives; for all \( n \)

\[
\bar{C}_R^{(n)}(\tau, 0, 0) = \left( \frac{\partial^n \bar{C}_R}{\partial \tau^n} \right)_{\tau = 0, R = 0} \sim [T - T_c(0)]^{-\alpha_n},
\]

where

\[
\alpha_n = \alpha_0 + n\varphi
\]

and \( \alpha_0 \) is the two-dimensional specific-heat index \( \alpha_0 = 0 \).

The same homogeneity arguments for the variable \( R \) applied to the pair correlation function make predictions for the second-moment function \( \mu_2 \):

\[
\mu_2^{(n)}(\tau, 0, 0) = \left( \frac{\partial^n \mu_2}{\partial \tau^n} \right)_{\tau = 0, R = 0} \sim [T - T_c(0)]^{-\nu_0 s_R e^\varphi},
\]

where \( \nu_0 = 1 \), \( \gamma_0 = -\frac{1}{4} \) as before, and \( \varphi \) must be the same crossover exponent as for thermodynamic functions by virtue of the fluctuation-dissipation theorem connecting the correlation function to the bulk susceptibility.

Abe and Suzuki first presented arguments implying \( \varphi = \gamma_0 \) based upon an additional assumption concerning "scaling" properties of multispin correlation functions. However, more recently \( \varphi \) has been proven rigorously equal to \( \gamma_0 \) without making the Abe-Suzuki assumptions. The latter work starts directly from the high-temperature expansion of the susceptibility and considers graphical contributions with one out-of-plane bond to show \( \bar{x}^{(1)} \sim \tau^{-\gamma_0} \). By Eq. (1.9) it follows that if a constant gap exponent exists it must equal \( \gamma_0 \).

Hence the general-\( R \) high-temperature series for \( \bar{x} \), \( \bar{C}_n \), and \( \mu_2 \) serve several worthwhile purposes with regard to the crossover problem. They are useful in determining whether a constant gap index \( \varphi \) exists at all, and, if it does, whether it checks with the expected value of \( \gamma_0 \). This provides a sensitive test of scaling in the parameter \( R \) for both thermodynamic functions and the two-spin correlation function. Previous work has been restricted to the existing general-\( R \) \( \bar{x}^{(s)} \) series, \( ^4 \) which have been analyzed\( ^5,10 \) by various techniques with the conclusions in dispute. In Paper II following, \( ^11 \) all six functions \( \bar{x}^{(s)}, \bar{C}_n^{(0)}, \bar{C}_n^{(1)}, \mu_2^{(0)}, \mu_2^{(1)} \) are considered, and it is concluded there is stronger evidence in favor of a constant gap exponent of \( \varphi = \gamma_0 = -\frac{1}{4} \).

II. METHOD OF CALCULATION

We use a computer program based upon the renormalized linked-cluster expansion theory of Wortis, Jasnow, and Moore. \( ^{18} \) The two-spin correlation function \( \bar{C}_2(\bar{T}) = \langle \delta_0 \delta_\pi(\bar{T}) \rangle - \langle \delta_0(\bar{T}) \rangle \langle \delta_\pi(\bar{T}) \rangle \) was expanded to tenth order in inverse temperature for a range of specific values for \( J_{xy} \) and \( J_x \) in the Hamiltonian (1.1).

From \( \bar{C}_2(\bar{T}) \), series for the reduced zero-field isothermal susceptibility

\[
\bar{x} = \bar{x}(J_{xy}, J_x) = \sum_\bar{T} \bar{C}_2(\bar{T}),
\]

the "second moment" of the correlation function

\[
\mu_2 = \mu_2(J_{xy}, J_x) = \sum_\bar{T} |\bar{T}|^2 \bar{C}_2(\bar{T}),
\]

and reduced specific heat
$$\bar{C}_H = \bar{C}_H(J_{xy}, J_z) = -\frac{1}{2} T \sum_r \bar{C}_d(\bar{r}) \tag{1.3}$$

were calculated. Here the coefficients of the respective high-temperature series depend upon the particular values of $J_{xy}$ and $J_z$ set at the beginning of the computer program. Given the coefficients for eleven different combinations of $J_{xy}$ and $J_z$, we were able to solve simultaneous linear equations to determine the general series coefficients for arbitrary values of these parameters to tenth order.

Tables I and II present coefficients $a_{nj}$ through $n = 10$ for the reduced susceptibility series

$$\bar{x} = \sum_{m=0}^{n} \sum_{j=0}^{m} a_{nj} \tanh^{m-j}(\beta J_{xy}) \tanh^j(\beta J_z), \tag{2.4}$$

with $\beta = 1/k_BT$ and the $a_{nj}$ integers related to a class of graphs on the lattice with $(h-j)$ bonds in the $x-y$ plane and $j$ bonds in the $z$ direction.\(^\text{13}\)

Tables III and IV present the corresponding integer coefficients $b_{nj}$ in the second-moment series:

$$\mu_z = \sum_{\bar{r}} |\bar{t}|^2 \bar{C}_d(\bar{r}) = \sum_{m=0}^{n} \sum_{j=0}^{m} b_{nj} \tanh^{m-j}(\beta J_{xy}) \tanh^j(\beta J_z). \tag{2.5}$$

From the double tanh expansions one may reexpand to obtain for the coefficient of $\beta^n$ a polynomial in $R$ through $R^n$.

While $\bar{x}$ and $\mu_z$ have contributions to tenth order from correlation functions to lattice points up to ten lattice spacings away from the origin, the specific heat has contributions from the correlation function to only the nearest neighbors of the origin. On the sc lattice, the form for $\bar{C}_H$ is

$$\bar{C}_H^{sc} = TC_H^{sc}/N = \sum_{m=0}^{n} \sum_{j=0}^{m} c_{nj} J_{xy}^{m-j} J_z^j \beta^{m-j} \tag{2.6}$$

since all odd $n$ coefficients vanish and only even

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE V.** Coefficients $c_{nj}$ in reduced specific heat series for the sc lattice,$$
\bar{C}_H = TC_H^{sc}/N = \sum_{m=0}^{n} \sum_{j=0}^{m} c_{nj} J_{xy}^{m-j} J_z^j \beta^{m-j} = \sum_{n=0}^{n} j \left( \sum_{j=0}^{m} c_{nj} R^j \right) \beta^{m-j},
$$
coefficients with $n$ or $j$ odd are zero.
powers of $R$ enter for the even $n$. For the fcc, odd-order $n$ are now nonzero as well:

$$C_H^{tec} = T C_H^{tec}/N = \sum_{n=2}^{\infty} \sum_{J_x} c_n J_{xy}^{n-n'} J_x^{n-1} \beta^{n-1}$$

$$= \sum_{n=2}^{\infty} J_{xy}^{n-n'} (c_{nR} R^n + c_{nS} R^n + \ldots + c_{nF} R^{n-n'}) \beta^{n-1},$$

where $n'=n$ if $n$ is even, $n'=n-1$ if $n$ is odd. Here $c_{n0}=0$, and again only even powers of $R$ contribute. The specific heat coefficients $c_n$ appear in Tables V and VI.

III. CHECKS ON SOLUTIONS

The series in the limits $R=0$ and $R=\infty$ are known, as is the isotropic $R=1$ case. For $R=0$, both the anisotropic sc and fcc lattices reduce to one-dimensional square lattices. The $R=\infty$ limit ($J_x=0$, $J_z$ finite) on the anisotropic sc gives a set of non-interacting one-dimensional linear chains, and on the fcc it gives a body-centered cubic (bcc) lattice.

We verified that the general series for $\tilde{\chi}$, $\mu_2$, and $C_H$ reproduced the expected known series in all cases. For example, reading down the "zeroth" column in any table gives the Ising-square-lattice series, while summing the numbers in a given row across corresponds to $R=1$ and yields the appropriate coefficient for the respective isotropic lattice (sc or fcc).\(^\dagger\) Reading the last entry in each row down a table checks the $R=\infty$ limits—for example, we immediately recognize the familiar linear chain results in the sc susceptibility table (cf. Table I).

Further checks on the tabulated numbers are provided by the very recent rigorous results of Liu and Stanley,\(^\dagger\) who show

$$\tilde{\chi}(1) = g \beta J_{xy}(\tilde{\chi}(0))^2,$$  \hspace{2cm} (3.1)

$$\mu_2(1) = g \beta J_{xy}[\tilde{\chi}(0)^2 + 2\tilde{\chi}(0)^2 \mu_2(0)],$$  \hspace{2cm} (3.2)

where $g$ is the number of out-of-plane nn bonds ($g=2$, 8 for the sc, fcc, respectively). Here, of course, $\tilde{\chi}(0)$ and $\mu_2(0)$ are the reduced susceptibility and second moment for the square lattice (obtained by reading down the zeroth column), and $\tilde{\chi}(1)$ and $\mu_2(1)$ are as defined in Eqs. (1.4) and (1.12). We have verified that the numbers in Tables I–IV satisfy Eqs. (3.1) and (3.2).

*Note added in proof. The reader will note that all the entries except the main diagonal and the first column of Tables I and III (sc lattice) are divisible by 8, while for Tables II and IV (fcc lattice) they are divisible by 16.

ACKNOWLEDGMENTS

We gratefully acknowledge helpful conversations with Dr. G. Paul and particularly L. Liu, who kindly assisted in some of the numerical checking procedures. We also wish to thank D. Karo, D. Lambeth, and M. H. Lee for discussions.

\(^*\)Work forms a portion of a Ph.D. thesis to be submitted by Fredric Harbus to the Physics Department of MIT. Work supported by NSF, ONR, and AFOSR.

\(^\dagger\)NSF Predoctoral Fellow.

Scaling with Respect to a Parameter for the Gibbs Potential and Pair Correlation Function of the $S=\frac{1}{2}$ Ising Model with Lattice Anisotropy*

Richard Krasnow, Fredric Harbus, † Luke L. Liu, and H. Eugene Stanley

Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 24 May 1972; revised manuscript received 30 August 1972)

I. INTRODUCTION

Interest has recently focused
d on magnetic model systems with different coupling strengths in different lattice directions ("lattice anisotropy") described by the Hamiltonian

$$\mathcal{H} = -J_x \sum_{\langle ij \rangle} s_i s_j - J_y \sum_{\langle ij \rangle} s_i s_j$$

and

$$\mathcal{N} = -J_x \sum_{\langle ij \rangle} s_i s_j + J_y \sum_{\langle ij \rangle} s_i s_j$$

thereby defining $R = J_x/J_y$ as the ratio of interplanar to intraplanar coupling strengths. Here $s_i = \pm 1$, the first sum is over nearest-neighbor (nn) spins in the xy plane, while the second sum is over spins whose relative displacement vector has a z component. The Hamiltonian (1.1) has previously been studied
to test the predictions of the "universal hypothesis,

and (b) to examine critical behavior upon crossing over from a three-dimensional to a two-dimensional lattice as $R \to 0$. Of particular interest is the "crossover exponent" $\psi$ giving the variation of critical temperature $T_\nu(R)$ with $R$ for small $R$,

$$T_\nu(R) = T_\nu(0) - R^{-\psi},$$

and its relation to various scaling predictions.

In the preceding paper
hereafter referred to as Paper 1), the reduced susceptibility $\tilde{\chi}$, the reduced specific heat $\tilde{C}_H$, and the second moment $\mu_2$ were defined, and high-temperature series for arbitrary $R$ were presented for these quantities on both the sc and fcc lattices. The implications of scaling of thermodynamic functions and of the pair correlation function with respect to the parameter $R$ were discussed. In particular, the consequences of assuming the Gibbs potential to be a generalized homogeneous function (GHF) of the variables $T = T_\nu(R)$

was mentioned in Ref. 10 without proof.