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Comparing null models for testing multifractality in time series

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Abstract – The behaviors of fat-tailed distribution, linear long memory, and nonlinear long memory are considered as possible sources of apparent multifractality. Which behavior should be preserved in null models plays an important role in statistical tests of empirical multifractality. In this paper, we compare the performance of two null models on testing the existence of multifractality in fractional Brownian motions (fBm), Markov-switching multifractal (MSM) model, and financial returns. One null model is obtained by shuffling the original data, which keeps the distribution unchanged. The other null model is generated by the iterative amplitude adjusted Fourier transform (IAAFT) algorithm, which insures that the surrogate data and the original data sharing the same distribution and linear long memory behavior. We find that the tests based on the shuffle null model only reject the multifractality in fBm with $H = 0.5$ and the tests based on the IAAFT null model reject the multifractality in fBms (except for $H = 0.1$). And the multifractality in MSM and financial returns are significantly supported by the tests based on both null models. Our findings also shed light on the necessity of choosing suitable null models to test multifractality in other complex systems.

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Introduction. – Multifractality in financial markets has attracted considerable research interest both from the economic and the physical perspective in the last twenty years. The contributions of the existing literature are briefly listed as follows (refer to the review [1] for more information).

Many methods are proposed to empirically uncover the multifractality, such as the partition function approach [2], the structure function approach [3], the wavelet transform approach [4–6], the detrended fluctuation approaches [7–9], multifractal natural time analysis [10] and so on. By employing the above-mentioned methods, it is found that many financial time series (returns, volatilities, bid-ask spreads, to list a few) from different markets around the world exhibit significant multifractal characteristics [11–20], which not only inspires people to construct models (multifractal random walk (MRW) [21,22], Markov-switching multifractal (MSM) models [23,24], and so on) to replicate such important stylized facts, but also motive researchers to find the sources of multifractality.

The behaviors of fat-tailed distributions and (linear and nonlinear) long-range dependence are considered as possible origins of multifractal characteristics in financial series [25–31]. However, many empirical results reveal that the series generated from monofractal models can produce spurious multifractality [32–36], which calls for strict statistical tests in empirical analysis of multifractality. As we know, the results of statistical tests on the empirical multifractality strongly depend on how we choose the null models. If the multifractality is only attributed to the fat tail distributions, statistical tests based on the null models of shuffled series may fail. Thus, it is important to compare the performance of different null models in testing multifractality.

In this paper, we compare the performance of two null models in testing the existence of multifractality in fractional Brownian motions (fBm), MSM model, and financial returns. One null model is obtained by shuffling the

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original data, which keeps the distribution unchanged. The other null model is generated by the IAAFT algorithm, which insures that the surrogate data and the original data sharing the same distribution and linear long-memory behavior. Our results show that an improper choice of null models will lead to incorrect conclusions.

Methods. – We describe here the mathematical models for data generation, whose fractal or multifractal properties are known, and the null models applied for statistical tests.

Models for data generation. Fractional Brownian motion and the Markov-switching multifractal model are employed to produce synthetic data. The fBm time series are generated by the MATLAB function "wfbm", corresponding to the algorithm proposed in ref. [37]. We only briefly introduce the MSM model here (refer to refs. [23,24] for details). The MSM model is defined as

$$r_i = \sigma^2 \prod_{k=1}^{\bar{k}} M_{k,i}\epsilon_i, \tag{1}$$

where ϵ_i are i.i.d. standard Gaussians and $M_{k,i}$ is the element of the volatility components $\{\overrightarrow{M_i}\}_{1\times\bar{k}}$ in period i. By assuming $\{\overrightarrow{M_i}\}_{1\times\bar{k}}$ following a first-order Markov process, each element $M_{k,i}$ renews with a probability of $\gamma_k = 1-(1-\gamma_{\bar{k}})^{b^{k-\bar{k}}}$, otherwise it remains unchanged, meaning $M_{k,i} = M_{k,i-1}$. The renewed value is drawn from a binomial distribution $[m_0, 1-m_0]$ with equal probability. In MSM, $\{\sigma^2, b, \gamma_{\bar{k}}, m_0\}$ are the parameters to be estimated and can be estimated by maximum likelihood estimation [24] and generalized method of moments [38]. $\bar{k}$ corresponds to the volatility frequency and is given while doing the estimation. For simplicity, we set the initial value $M_{k,0}$ as m_0.

Null models. We employ two null models to implement the statistical tests. One null model is obtained by shuffling the original series, which keeps the distribution unchanged. The other null model is produced by the iterative amplitude adjusted Fourier transform (IAAFT) algorithm [39], which preserves the same distribution and linear long-memory behavior as the original data. For a given series $\{x(t)\}$, the IAAFT algorithm is implemented as follows:

Step 1: We sort the series $\{x(t)\}$ in ascending order and denote it as $\{y_N\}$ and also perform the Fourier transform on $\{x(t)\}$ to give the squared amplitudes $\{Y_k^2\}$.

Step 2: We shuffle $\{x(t)\}$ and obtain an initial sequence $\{y_N^{(0)}\}$ for further iterations.

Step 3: We take the Fourier transform on $\{y_N^{(0)}\}$ and obtain the squared amplitudes $\{Y_k^{2,(0)}\}$. By replacing $\{Y_k^{2,(0)}\}$ with $\{Y_k^2\}$, we transformed back through the inverse Fourier transform. The resulting series is replaced by $\{x(t)\}$ with ranking ordering, denoted as $\{y_N^{(1)}\}$.

Step 4: If $\frac{\sum(Y_k^{2,(0)}-Y_k^2)^2}{\sum Y_k^2} < 10^{-6}$, the iteration stops; otherwise, let $\{y_N^{(0)}\} = \{y_N^{(1)}\}$ and repeat Step 3.

Statistical tests. Following the studies on investigating the sources of multifractality [28,29], we define the MF statistic as the singularity width $\Delta\alpha$ of the multifractal spectrum. For the synthetic data, we perform the statistical tests as follows:

Step 1: For a given model we generate a series of synthetic data with a size of 2^{16}.

Step 2: The multifractal analysis is performed on the synthetic data by means of MF-DFA. The MF statistic $\Delta\alpha_{\mathrm{ORIG}}$ is estimated.

Step 3: The synthetic series is shuffled to remove any potential correlations. The same multifractal analysis is conducted on the shuffled series and the MF statistic $\Delta\alpha_{\mathrm{SHUF}}$ is determined.

Step 4: The surrogate series is obtained by injecting the synthetic series into the IAAFT algorithm. The same multifractal analysis is carried out on the surrogate series and the MF statistic $\Delta\alpha_{\mathrm{IAAFT}}$ is calculated.

These steps are repeated until we accumulate 10000 sets of $\{\Delta\alpha_{\mathrm{ORIG}}$, $\Delta\alpha_{\mathrm{SHUF}}$, and $\Delta\alpha_{\mathrm{IAAFT}}\}$. We then use the two-sample Kolmogorov-Smirnov (KS) test to check the difference between $\Delta\alpha_{\mathrm{ORIG}}$ and $\Delta\alpha_{\mathrm{SHUF}}$ (or $\Delta\alpha_{\mathrm{IAAFT}}$). Note that the moment orders q vary from -4 to 4 with a step of 0.1 and the linear fits are employed as the detrending filter in MF-DFA analysis. The scaling range is set as $[2^4, 2^{12}]$ for fBM, $[2^4, 2^{12}]$ for MSM, and $[2^4, 2^9]$ for Dow Jones Industrial Average (DJIA) returns when estimating the generalized Hurst indexes.

The statistical tests on financial data are very similar to the tests on the synthetic data, but differ in two aspects. One is that the set $\{\Delta\alpha_{\mathrm{ORIG}}\}$ contains 10000 points for synthetic data, but only a value for financial data. The other is that the algorithms of shuffling and IAAFT are performed on each realization for synthetic data, but always on the same series for financial data.

Results. – Here, we present the results of statistical tests on time series generated from mathematical models.

Monofractal models. We first perform our statistical tests on the monofractal data to test the absence of multifractality. The monofractal data are fBm series with Hurst indexes H varying from 0.1 to 0.9 with a step of 0.1. Theoretically, there should be no multifractality in fBms by definition. However, in empirical analysis the multifractality of fBms usually approaches vanishing, but is never

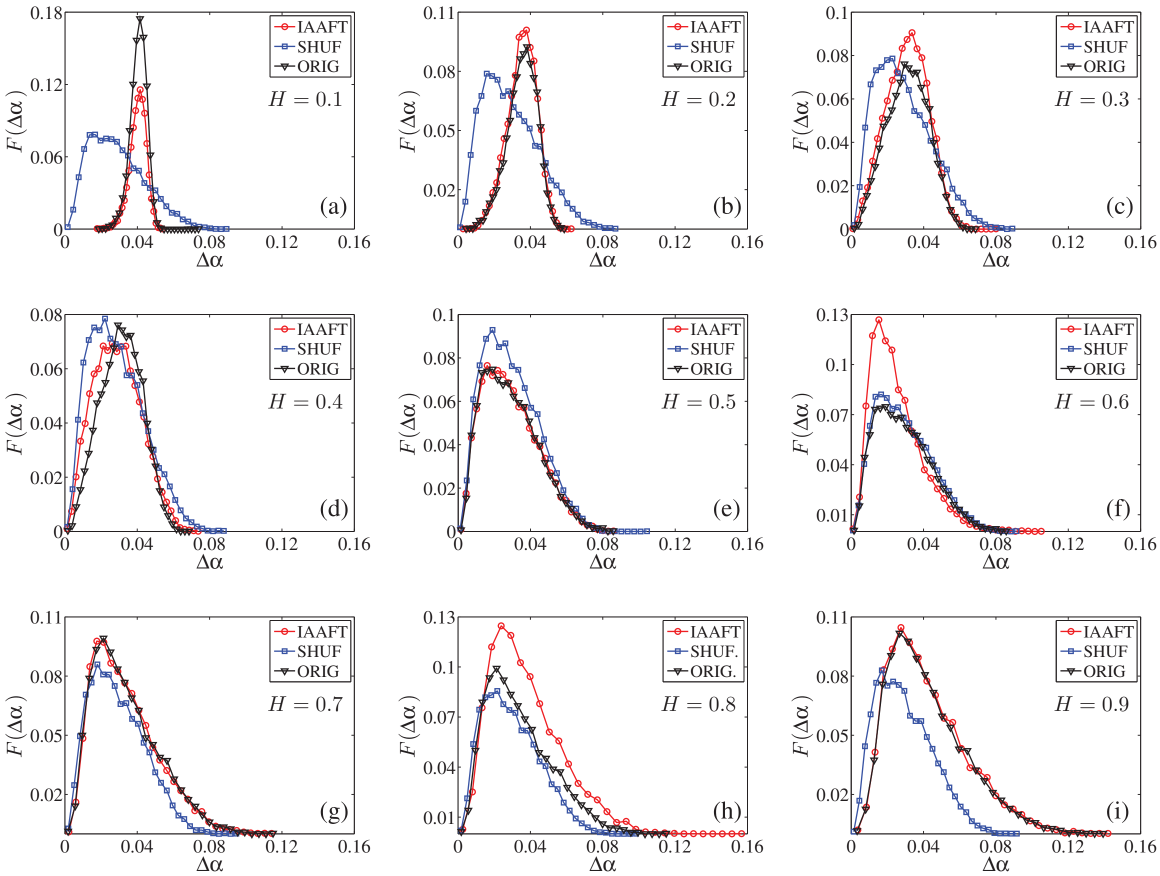

Fig. 1: Results of statistical tests on fBm for different Hurst indexes H. For each H we generate 10000 realizations and for each realization we generate one shuffled series and one IAAFT series.We estimate the MF statistics for each realization and its shuffled and IAAFT series and thus obtain 10000 values of $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, $\Delta\alpha_{\rm IAFFT}$. The frequency of $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ are plotted in panels (a)–(i) for different H. (a) $H = 0.1$. (b) $H = 0.2$. (c) $H = 0.3$. (d) $H = 0.4$. (e) $H = 0.5$. (f) $H = 0.6$. (g) $H = 0.7$. (h) $H = 0.8$. (i) $H = 0.9$.

vanishing. In other words, the MF statistic $\Delta\alpha$ of fBms should be a small number and close to zero.

As we know, the long-memory behaviors and fat-tailed distributions are considered as sources of empirical multifractality. Thus, we can infer that: 1) for fBms the empirical multfractality is purely determined by their memory behaviors and the surrogate series generated by the IAAFT algorithm should have the same multifractality as the original fBms; 2) the stronger memory behaviors the fBms have, the greater the resulting empirical multifractality is. Thus, fBms with Hurst index $H = 0.5$ have the weakest empirical multifractality. As the shuffled series do not have any correlated behaviors, they coincidentally correspond to the case of fBm with Hurst indexes $H = 0.5$. This means that the shuffled fBm should also have the weakest empirical multifractality.

To have an overview of the testing results on fBm, we illustrate the frequency of the spectral widths $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ for different H in fig. 1. We can see that the spectral width $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ are not vanishing but close to zero, evidenced by the narrow span of $\Delta\alpha$ in each panel. Such results do chime with the monofractal nature of fBms. It is observed that the frequency curves of $F(\Delta\alpha_{\rm ORIG})$ and $F(\Delta\alpha_{\rm IAAFT})$ are nearly overlapping together, indicating that the original and IAAFT series exhibit the same characteristic of empirical multifractality. Furthermore, one can see that in all panels the $F(\Delta\alpha_{\rm SHUF})$ curves share the same pattern and have the same spans both on the X-axis and on the Y-axis, because any correlated behaviors in shuffled series are removed and all shuffled series correspond to the case of fBm with $H = 0.5$. This can also explain the observation that the three frequency plots $F(\Delta\alpha_{\rm ORIG})$, $F(\Delta\alpha_{\rm SHUF})$, and $F(\Delta\alpha_{\rm IAAFT})$ are very close to each other in panel (e).

The frequency plots in fig. 1 only visually present an impression that the testing results based on two null models are roughly consistent with theoretical arguments. In the following, we will quantitatively describe such consistence. The mean and standard deviations of the MF statistics $\Delta\alpha$ are listed in panel A of table 1. As mentioned above, we expect that 1) the MF statistic $\Delta\alpha$ is close to zero, supported by the evidence that the average values of the three MF statistics are very small numbers for different Hurst indexes; 2) the fBm with $H = 0.5$ and the shuffled series

Table 1: Comparison of the statistical tests on fBm based on two null models. Panel A lists the mean and standard deviation of the MF statistics of the fMb series, the shuffled series, and the IAAFT series. The numbers in parentheses are the standard deviations. Panel B (respectively, C) presents the results of the two-sample KS test between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$ (respectively, between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm IAAFT}$). The null hypothesis is H_0: $C(\Delta\alpha_{\rm ORIG}) = C(\Delta\alpha_{\rm SHUF})$ (respectively, H_0: $C(\Delta\alpha_{\rm ORIG}) = C(\Delta\alpha_{\rm IAAFT})$) and the alternative hypothesis is H_1: $C(\Delta\alpha_{\rm ORIG}) < C(\Delta\alpha_{\rm SHUF})$ (respectively, H_1: $C(\Delta\alpha_{\rm ORIG}) < C(\Delta\alpha_{\rm IAAFT})$). $C(x)$ represents the cumulative distribution of x and $C(x_1) < C(x_2)$ means that the data in x_1 is larger than the data in x_2. The values in square brackets correspond to the p-values of the KS test.

	$H=0.1$	$H=0.2$	$H=0.3$	$H=0.4$	$H=0.5$	$H=0.6$	$H=0.7$	$H=0.8$	$H=0.9$
Panel A: Mean and standard deviation of $\Delta\alpha$									
$\Delta\alpha_{\rm ORIG}$	0.041	0.035	0.031	0.031	0.029	0.029	0.034	0.034	0.042
	(0.005)	(0.008)	(0.011)	(0.011)	(0.015)	(0.015)	(0.018)	(0.018)	(0.022)
$\Delta\alpha_{\rm SHUF}$	0.029	0.029	0.029	0.029	0.029	0.029	0.029	0.029	0.029
	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)
$\Delta\alpha_{\rm IAAFT}$	0.040	0.035	0.031	0.029	0.029	0.025	0.034	0.038	0.042
	(0.005)	(0.008)	(0.011)	(0.013)	(0.015)	(0.014)	(0.018)	(0.019)	(0.022)
Panel B: Two-sample KS test between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$									
KS stats.	0.582	0.340	0.157	0.033	0.003	0.035	0.093	0.148	0.227
	[0.000]	[0.000]	[0.000]	[0.000]	[0.890]	[0.000]	[0.000]	[0.000]	[0.000]
Panel C: Two-sample KS test between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm IAAFT}$									
KS stats.	0.030	0.012	0.012	0.011	0.010	0.004	0.012	0.004	0.007
	[0.000]	[0.259]	[0.253]	[0.304]	[0.367]	[0.831]	[0.225]	[0.872]	[0.629]

have the weakest empirical multifractality, evidenced by the observations that $\Delta\alpha_{\rm ORIG}$ exhibit a U-shape pattern with a valley at $H = 0.5$ and $\Delta\alpha_{\rm SHUF}$ remains a constant, equaling the valley of $\Delta\alpha_{\rm ORIG}$; and 3) the fBm and its IAAFT surrogates have the same characteristic of empirical multifractality, confirmed by the result that the mean of $\Delta\alpha_{\rm IAAFT}$ equals the mean of $\Delta\alpha_{\rm ORIG}$ when H is fixed.

By employing the two-sample KS test, we further check whether the MF statistic pairs ($\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$, $\Delta\alpha_{\rm ORIG}$ and $\Delta_{\rm IAAFT}$) are from the same cumulative distribution. Panel B reports the results of KS tests between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$. The null hypothesis is that $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$ follow the same cumulative distribution and the alternative hypothesis is that the cumulative distribution of $\Delta\alpha_{\rm ORIG}$ is smaller than that of $\Delta\alpha_{\rm SHUF}$, meaning $\Delta\alpha_{\rm ORIG} > \Delta\alpha_{\rm SHUF}$. At the significant level of 1%, the null hypothesis cannot be rejected only for $H = 0.5$ and for the remaining Hurst indexes, we reject the null hypothesis and accept the alternative hypothesis, indicating that the shuffling series is not a good null model for testing the absence of multifractality in monofractal model. Panel C presents the two-sample KS tests between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm IAAFT}$. One can see that only for $H = 0.1$, the null hypothesis is rejected, implying that the IAAFT surrogate is a good null model for statistically testing the absence of multifractality in the monofractal process.

Statistical tests on multifractal models. We utilize the MSM models to simulate multifractal series and the four model parameters are fixed as $m_0 = 1.5$, $\sigma = 0.5$, $b = 3$, and $\gamma_{\bar{k}} = 0.95$, which are also used to generate multifractal data in ref. [24]. The volatility frequency $\bar{k}$ takes one of the six values: $\bar{k} \in \{2, 5, 8, 10, 15, 20\}$.

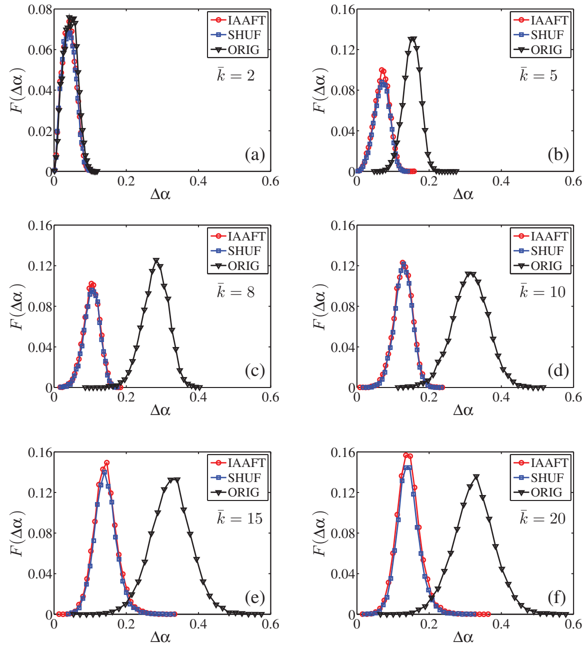

Fig. 2: Results of statistical tests on MSM for different volatility frequencies $\bar{k}$. For each $\bar{k}$ we generate 10000 realizations and for each realization we generate one shuffled series and one IAFFT series. We estimate the MF statistics for each realization and its shuffled and IAAFT series and thus obtain 10000 values of $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$. The frequency of $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ are plotted in panels (a)–(i) for different $\bar{k}$. (a) $\bar{k} = 2$. (b) $\bar{k} = 5$. (c) $\bar{k} = 8$. (d) $\bar{k} = 10$. (e) $\bar{k} = 15$. (f) $\bar{k} = 20$.

As the MSM model is proposed to model financial returns, we can infer that the Hurst index of MSM should be very close to 0.5. By estimating the mean and standard

Table 2: Comparison of the statistical tests on MSM based on two null models. Panel A lists the mean and standard deviation of the MF statistics of the MSM series, the shuffled series, and the IAAFT series. The numbers in parentheses are the standard deviations. Panel B (respectively, C) presents the results of the two-sample KS test between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$ (respectively, between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm IAAFT}$). The null hypothesis is H_0: $C(\Delta\alpha_{\rm ORIG}) = C(\Delta\alpha_{\rm SHUF})$ (respectively, H_0: $C(\Delta\alpha_{\rm ORIG}) = C(\Delta\alpha_{\rm IAAFT})$) and the alternative hypothesis H_1: $C(\Delta\alpha_{\rm ORIG}) < C(\Delta\alpha_{\rm SHUF})$ (respectively, H_1: $C(\Delta\alpha_{\rm ORIG}) < C(\Delta\alpha_{\rm IAAFT})$). $C(x)$ represents the cumulative distribution of x and $C(x_1) < C(x_2)$ means that the data in x_1 is larger than the data in x_2. The values in square brackets correspond to the p-values of the KS test.

	$\bar{k}=2$	$\bar{k}=5$	$\bar{k}=8$	$\bar{k}=10$	$\bar{k}=15$	$\bar{k}=20$
Panel A: Mean and standard deviation of $\Delta\alpha$						
$\Delta\alpha_{\rm ORIG}$	0.046	0.152	0.281	0.313	0.324	0.324
	(0.019)	(0.023)	(0.036)	(0.050)	(0.056)	(0.057)
$\Delta\alpha_{\rm IAAFT}$	0.042	0.069	0.104	0.128	0.143	0.143
	(0.018)	(0.021)	(0.023)	(0.026)	(0.032)	(0.032)
$\Delta\alpha_{\rm SHUF}$	0.042	0.069	0.104	0.128	0.142	0.142
	(0.018)	(0.021)	(0.022)	(0.025)	(0.031)	(0.031)
Panel B: Two-sample KS test between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm SHUF}$						
KS stat.	0.087	0.942	0.997	0.986	0.956	0.954
	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
Panel C: Two-sample KS test between $\Delta\alpha_{\rm ORIG}$ and $\Delta\alpha_{\rm IAAFT}$						
KS stat.	0.086	0.940	0.997	0.985	0.953	0.950
	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]

deviations of the Hurst indexes of 10000 simulated MSM series for different frequency $\bar{k}$, we obtain that the mean Hurst indexes of the six frequencies all equal to 0.50 ± 0.01, confirming no linear correlated behaviors in MSM. Thus, one can expect that the null models of shuffled series and IAAFT series will have very similar results.

Figure 2 illustrates the frequency plots of MF statistics for the MSM series and their shuffled and IAAFT series. The behavior of no linear correlation in MSM is further corroborated by the observation that the curves of $\Delta\alpha_{\rm SHUF}$ and $\Delta\alpha_{\rm IAAFT}$ are all overlapping together in panels (a)–(f).

As metioned above, the possible sources of the multifractality in MSM are the fat-tailed distribution and the nonlinear long memory. If the multifractality only originates from the fat-tailed distribution, one may expect that the data generated by both null models should exhibit the same multifractality as the original data. Figure 2 illustrates the frequency plots of MF statistics for different frequency $\bar{k}$. In panel (a), we find that the frequency plots of $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ almost overlap together, consistently with the equivalence of the average values of $\Delta\alpha_{\rm ORIG}$, $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ for $\bar{k}=2$ in panel A of table 2. The other panels present that there is a significant deviation between the curves of the original MSM series and those of the null models, indicating that the nonlinear long memory also plays an important role in the multifractal nature of MSM. We also find that the volatility frequency $\bar{k}$ has a strong influence on the multifractal strength of MSM, as it is observed that the frequency plots of $\Delta\alpha_{\rm ORIG}$ depart from the curves of $\Delta\alpha_{\rm SHUF}$ and $\Delta\alpha_{\rm IAAFT}$ towards to positive infinity when the frequency $\bar{k}$ increases from 2 to 15 in panels (a)–(e) of fig. 2, and the average value of $\Delta\alpha_{\rm ORIG}$ increases with the increment of $\bar{k}$ when $\bar{k} \leq 15$ in panel A of table 2. One can also notice that the values of $\Delta\alpha_{\rm SHUF}$ and $\Delta\alpha_{\rm IAFFT}$ are all significantly greater than 0. The possible explanation is that our simulated series length 2^{16} is not long enough to reach the convergence of $\Delta\alpha$ for the shuffled fat-tailed MSM series [40]. However, increasing $\bar{k}$ cannot infinitely increase the strength of multifractality for MSM models, because the distance between the curves of null models and original MSM series almost saturates when the frequency $\bar{k}$ approaches 15, evidenced by the plots in panels (e) and (f) and the equation $\Delta\alpha_{\rm ORIG}^{\bar{k}=15} = \Delta\alpha_{\rm ORIG}^{\bar{k}=20}$ in table 2.

Two-sample KS tests are also conducted between the MF statistics of the original MSM series and the null models. The results are listed in panels B and C of table 2. One can see that the null hypothesis can be rejected at the significant level of 0.01 for all tests, indicating that the alternative hypotheses $C(\Delta\alpha_{\rm ORIG}) < C(\Delta\alpha_{\rm SHUF})$ and $C(\Delta\alpha_{\rm ORIG}) < C(\Delta\alpha_{\rm IAAFT})$ are favored. Thus, we have $\Delta\alpha_{\rm ORIG} > \Delta\alpha_{\rm SHUF}$ and $\Delta\alpha_{\rm ORIG} > \Delta\alpha_{\rm IAAFT}$, suggesting that the tests based on both null models support the existence of multifractality in MSM. According to the results in tables 1 and 2, we can further infer that there are three components in the multifractal width $\Delta\alpha_{\rm ORIG}$ in the MSM for $\bar{k} \geq 15$, such that 1) one systematic constant component of 0.03 ($\Delta\alpha_{\rm SHUF}$ in table 1) probably due to the finite-size effects, 2) one components of 0.18 ($\Delta\alpha_{\rm ORIG} - \Delta\alpha_{\rm SHUF}$ in table 2) attributed to the nonlinear correlation, and 3) the rest of 0.11 may be due to the fat-tailed distribution.

Application to financial series. – We further perform statistical tests to check the multifractality in financial returns based on the two null models. The daily Dow

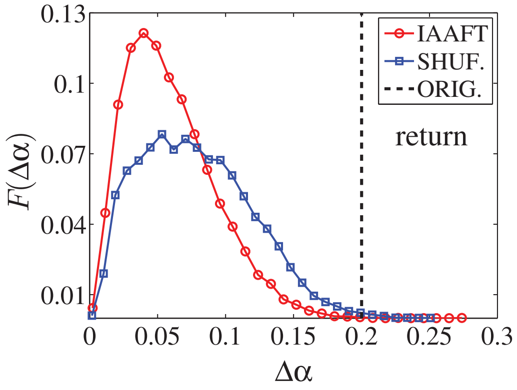

Fig. 3: Results of statistical tests on DJIA returns. We estimate their MF statistic $\Delta\alpha_{\rm ORIG}$, and generate 10000 shuffled series and 10000 IAAFT series. The MF statistics $\Delta\alpha_{\rm SHUF}$, and $\Delta\alpha_{\rm IAFFT}$ for these shuffled and IAAFT series are estimated. The frequency of $\Delta\alpha_{\rm SHUF}$ and $\Delta\alpha_{\rm IAFFT}$ are plotted in panels (a), (b). The MF statistic $\Delta\alpha_{\rm ORIG}$ is illustrated as a vertical dashed line.

Table 3: Comparison of statistical tests on the multifractality in financial returns based on two null models. One is obtained by shuffling the original series and the other is generated by the IAAFT algorithm. The MF statistics of the original series, shuffled series, and IAAFT series and the p-value of the two null models are listed. The numbers in parentheses are the standard deviations.

	$\Delta\alpha_{\rm ORIG}$	$\Delta\alpha_{\rm SHUF}$	$\Delta\alpha_{\rm IAAFT}$	$p_{\rm SHUF}$	$p_{\rm IAAFT}$
return	0.200	0.078	0.060		
		(0.041)	(0.041)	[0.004]	[0.001]

Jones Industrial Average (DJIA) index is used to calculate the returns. The daily return is defined as the logarithmic difference of the daily closing price:

$$r(t) = \ln I(t) - \ln I(t-1), \tag{2}$$

where $I(t)$ is the closing price of the DJIA on day t. The spanning period of DJIA indexes is from 16 February 1885 to 17 June 2016, containing 36048 data points in total.

We estimate the MF statistics of the original, shuffled, and IAAFT returns and plot their frequencies in fig. 3(a). The vertical dashed line is the MF statistic of the original returns and the two frequency curves correspond to the MF statistics ($\Delta\alpha_{\rm SHUF}$ and $\Delta\alpha_{\rm IAAFT}$) of the shuffled and IAAFT returns. One can see that $F(\Delta\alpha_{\rm SHUF})$ is flat and wide and $F(\Delta\alpha_{\rm IAAFT})$ is sharp and narrow. We also list the MF statistic of the original, shuffled, and IAAFT returns in table 3. Although both curves differ greatly from each other, they all locate on the left side of the dashed line, consistently with $\Delta\alpha_{\rm ORIG} > \langle\Delta\alpha_{\rm SHUF}\rangle > \langle\Delta\alpha_{\rm IAAFT}\rangle$ in table 3. This provides direct evidence in favor of multifractality in DJIA returns. We thus estimate the p-value, corresponding to the null hypothesis H_0: $\Delta\alpha_{\rm ORIG} \leq \Delta\alpha_{\rm NULL}$. Table 3 reports the p-value of the returns for both null models and we find that both p-values are less than 0.01, indicating that the null hypotheses are significantly rejected at the level of 0.01. Our results reveal that the statistical tests based on both null models statistically support the existence of multifractality in DJIA returns, consistently with the testing results of MSM.

Conclusions. – In summary, we have performed statistical tests on the empirical multifractality of the synthetic data by means of two null models, generated by the algorithm of shuffling and IAAFT. We use monofractal (fBm) and multifractal (MSM) models to generate synthetic data. For fBm with H varying from 0.1 to 0.9 with a step of 0.1, the tests based on the shuffled data *wrongly* support the existence of multifractality except for $H = 0.5$ and the tests based on the IAAFT data reject the existence of multifractality except for $H = 0.1$, indicating the good performance of IAAFT while testing the spurious multifractality in monofractal process. For MSMs with $\bar{k} \in \{2, 5, 8, 10, 15, 20\}$, the tests based on both null models cannot reject the multifractal nature and the two null models are equivalent due to the fact that there is no linear correlations in the MSM time series (that is, $H = 0.5$). We also perform the same statistical tests on the DJIA returns. For returns, we obtain the same results as the MSM model, the statistical tests based on both null models favor the existence of multifractality.

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