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Understanding the cubic and half-cubic laws of financial fluctuations

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Abstract

Recent empirical research has uncovered regularities in financial fluctuations. Those are: (i) the cubic law of returns: returns follow a power law distribution with exponent 3; (ii) the half cubic law of volumes: volumes follow a power law distribution with exponent $\frac{3}{2}$; (iii) Approximate cubic law of number of trades: the number of trades in a given time intervals follows a power law distribution with exponent around 3. We discuss a new theory that explains them, as well as some related facts.

1. Some puzzling facts that require an explanation

We discuss some puzzling empirical facts in financial markets. In a separate paper [1,2], we will propose a theory to explain them. Those facts concern the distribution of returns, volume, and number of trades (see Ref. [3] for a partial review).

1.1. The cubic law of returns

Returns are found to have a power law distribution with an exponent of 3 [3–6]. This is true at horizon of 15 min to 1 week. More precisely, the distribution of return follows:

$$P(r > x) \sim \frac{1}{x^{\zeta_r}} \quad \text{with} \quad \zeta_r \approx 3 , \tag{1}$$

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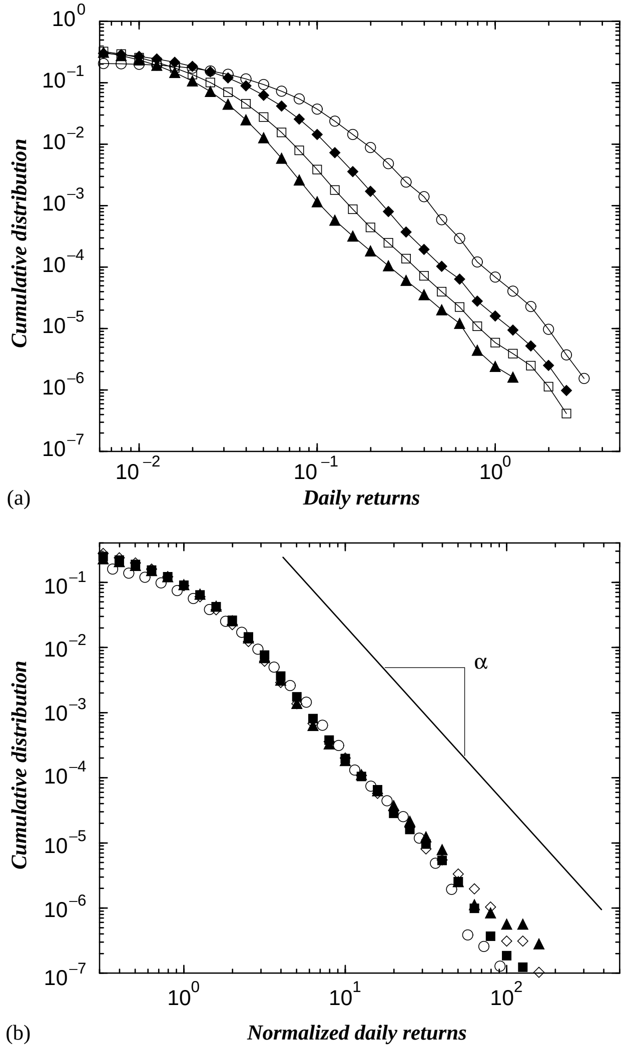

Fig. 1. (a) Cumulative distribution of the conditional probability $P(r > x)$ of the returns for companies with starting values of market capitalization S for $\Delta t = 1$ day from the CRSP database. We define uniformly spaced bins on a logarithmic scale and show the distribution of returns for the bins, $S \in (10^5, 10^6]$, $S \in (10^6, 10^7]$, $S \in (10^7, 10^8]$, $S \in (10^8, 10^9]$. (b) Cumulative conditional distributions of returns normalized by the average volatility σ_S of each bin. The plots collapsed to an identical distribution, with $\alpha = 2.70 \pm 0.10$ for the negative tail, and $\alpha = 2.96 \pm 0.09$ for the positive tail. Adapted from Ref. [13].

and we say that the "power law exponent" of returns is $\zeta_r \approx 3$. The smaller the interval Δt in which the returns are measured, the more data we have, and the closer to 3 is the measure of ζ_r. This can be seen in Fig. 1. Stocks of different sizes have different distributions, as small stocks have a larger volatility $var(r_t)$. But rescaling by the

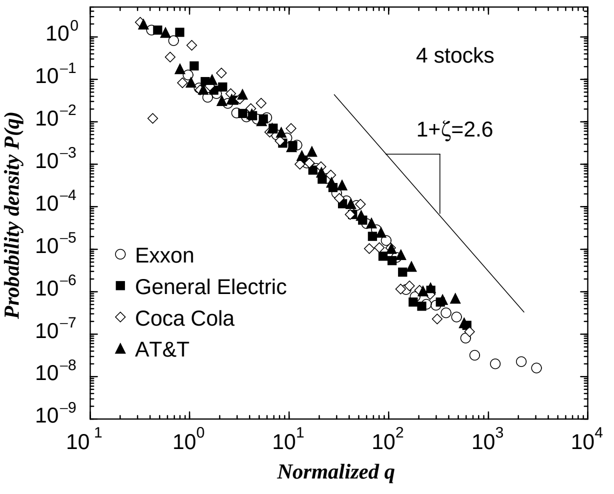

Fig. 2. Probability density function of the number of shares q traded (expressed as a fraction of the number of outstanding shares) normalized by the average value, for all transactions for the same four actively traded stocks. We find an asymptotic power-law behavior characterized by an exponent ζ_q. Fits yield values $\zeta_q = 1.87 \pm 0.13,\ 1.61 \pm 0.08,\ 1.66 \pm 0.05,\ 1.47 \pm 0.04$, respectively for each of the four stocks. Adapted from Ref. [9].

standard deviation reveals that they follow a common distribution (which suggests the term of "universality"). Crucially, this distribution has power law tails with a slope of 3. This slope of 3 seems to hold in different stock markets.

Importantly, the fact that $\zeta_r > 2$ rejects the possibility that asset returns follow a Levy distribution, a hypothesis advanced by Mandelbrot in a series of very influential papers [7,8].

1.2. The half-cubic law of trading volume

We find [9], that the density satisfies $f(q) \sim q^{-2.5}$, i.e., that the distribution is:

$$P(q > x) \sim \frac{1}{x^{\zeta_q}} \quad \text{with} \quad \zeta_q \approx \frac{3}{2} \,. \tag{2}$$

Fig. 2 illustrates this. Related results have also been found by Maslov and Mills [10].

1.3. Approximate cubic law of number of trades

We can do the same for the number of trades, and find [11]:

$$P(N > x) \sim \frac{1}{x^{\zeta_N}} \quad \text{with} \quad \zeta_N \approx 3.4 \,. \tag{3}$$

Again, those scalings seem to be stable across different types of stocks, different time periods and time horizons etc.

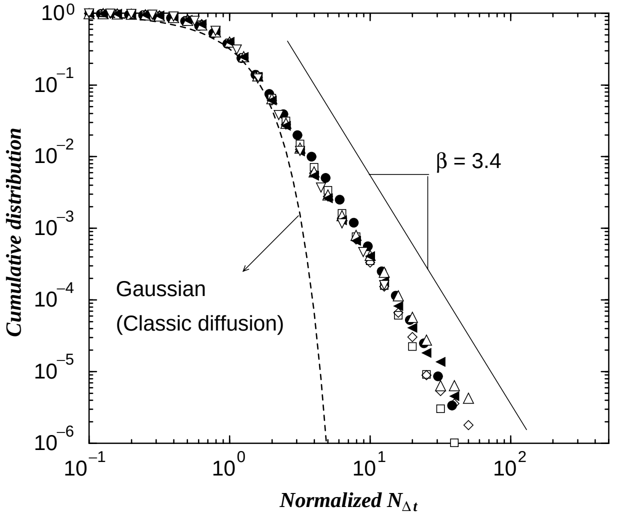

Fig. 3. Cumulative distribution of the normalized number of transactions $n_{\Delta t} = N_{\Delta t}/\langle N_{\Delta t}\rangle$. Each symbol shows the cumulative distribution $P(n_{\Delta t} > x)$ of the normalized number of transactions $n_{\Delta t}$ for all stocks in each bin of stocks sorted according to size. An analysis of the exponents obtained by fits to the cumulative distributions of each of the 1000 stocks yields an average value $\zeta_n = \beta = 3.40 \pm 0.05$.

Cumulative distribution of the normalized number of transactions $n_{\Delta t} = N_{\Delta t}/\langle N_{\Delta t}\rangle$. Each symbol shows the cumulative distribution $P(n_{\Delta t} > x)$ of the normalized number of transactions $n_{\Delta t}$ for all stocks in each bin of stocks sorted according to size. An analysis of the exponents obtained by fitting the cumulative distributions of each of the 1000 stocks yields an average value $\zeta_N = \beta = 3.40 \pm 0.05$ (Fig. 3).

2. On a possible explanation

A theory should explain not why there are power law fluctuations, but the precise values of the exponents. It should make testable "out of sample" predictions. Solomon and Richmond [12] present an interesting model. Gabaix et al. [1,2] proposes an alternative theory to explain those regularities [13,14]. This theory is based on economic optimization by heterogeneous agents. We believe that our theory satisfies these requirements. This theory makes a series of other predictions, most of which remain to be tested empirically. We have tested some of them, and the results were supportive. Hence the cubic laws of asset returns may have an explanation. The untested predictions of the theory could provide a guideline for a deeper exploration of scaling relations in the near future. The next few years of research, guided by the challenge of understanding the origins of the cubic laws of asset returns, will determine if the scaling relations predicted by the theory are empirically valid.

Acknowledgements

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