ELSEVIER

Physica A 266 (1999) 55–61

PHYSICA A

Distribution of shortest paths in percolation

Nikolay V. Dokholyan[a,*], Sergey V. Buldyrev[a], Shlomo Havlin[b], Peter R. King[c,d], Youngki Lee[a], H. Eugene Stanley[a]

[a] *Center for Polymer Studies, Physics Department, Boston University, Boston, MA 02215, USA*
[b] *Minerva Center and Department of Physics, Bar-Ilan University, Ramat Gan, 52900 Israel*
[c] *BP Amoco Exploration Operating Company Ltd., Sunbury-on-Thames, Middx., TW16 7LN, UK*
[d] *Department of Engineering, Cambridge University, Cambridge, UK*

Abstract

We present a scaling Ansatz for the distribution function of the shortest paths connecting any two points on a percolating cluster which accounts for (i) the effect of the finite size of the system, and (ii) the dependence of this distribution on the site occupancy probability p. We present evidence supporting the scaling Ansatz for the case of two-dimensional percolation.

PACS: 64.60.A

Keywords: Scaling Ansatz; Percolation; Finite-size effect

1. Introduction

The *chemical distance or minimal path*, ℓ, between two sites is defined as the shortest path on a percolating cluster connecting the two sites (Fig. 1). The quantity of interest here is the conditional probability, $P(\ell|r)$, that two sites taken from the same cluster, separated by geometrical distance r, are ℓ chemical distance away. The main questions are (i) to determine the dependence of this probability $P(\ell|r)$ on the finite size of the system L and (ii) the behavior of $P(\ell|r)$ in the off-critical regime ($p \neq p_c$).

The motivation for this study comes from the fact that in many realistic problems where the disordered media controls a transport process, dynamic properties such as conductivity and diffusion can be expressed in terms of chemical distance, in case of loopless aggregates, or for aggregates for which loops can be neglected [1]. For example, in oil recovery the first passage time from the injection well to a production well a distance r away is related to $P(\ell|r)$ (see [2]).

* Corresponding author. Fax: +1-617-353-9393; e-mail: dokh@polymer.bu.edu.

0378-4371/99/$ – see front matter
PII: S0378-4371(98)00574-3

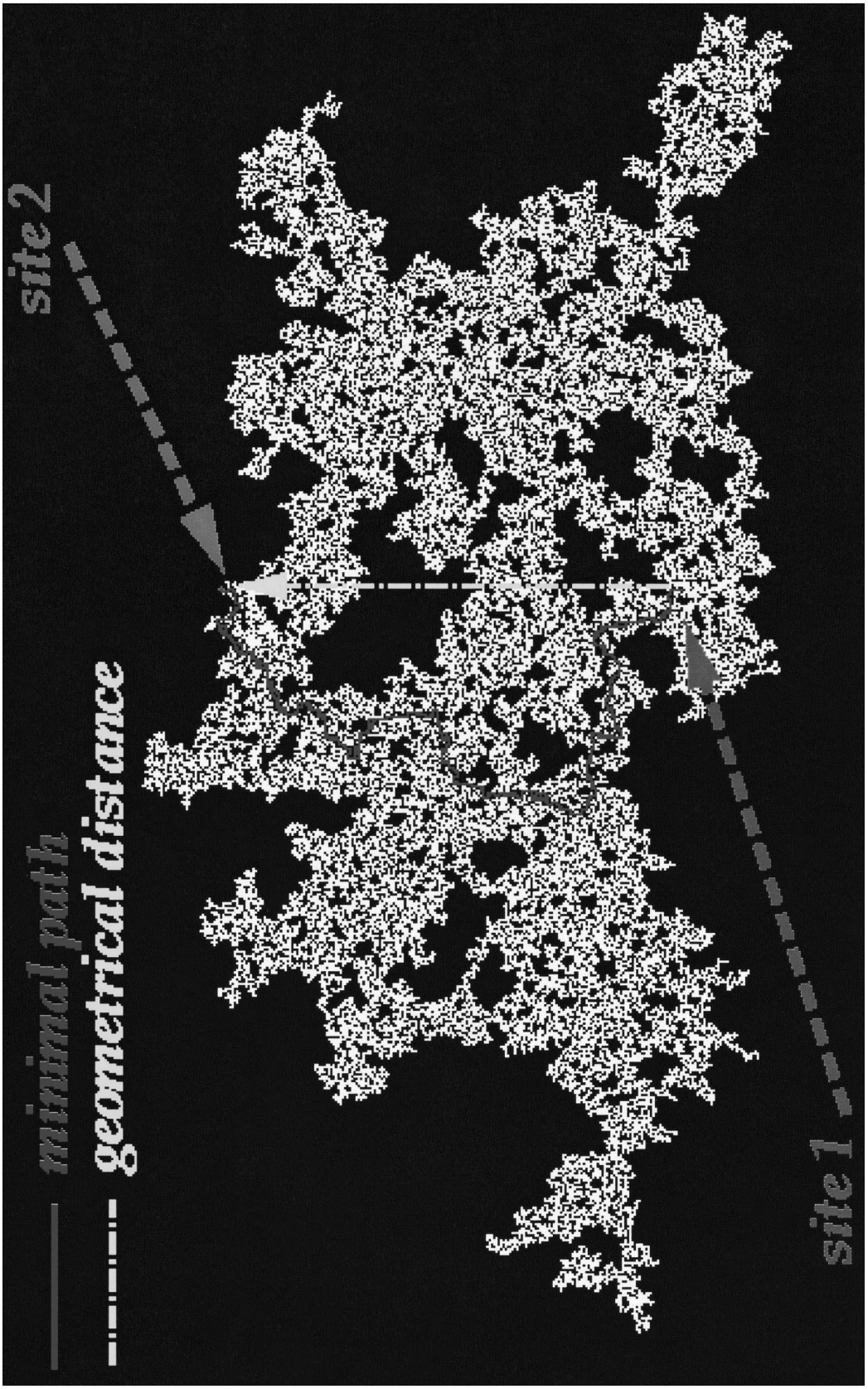

Fig. 1. The minimal path between two sites is defined as the shortest path on a percolating cluster connecting the two sites.

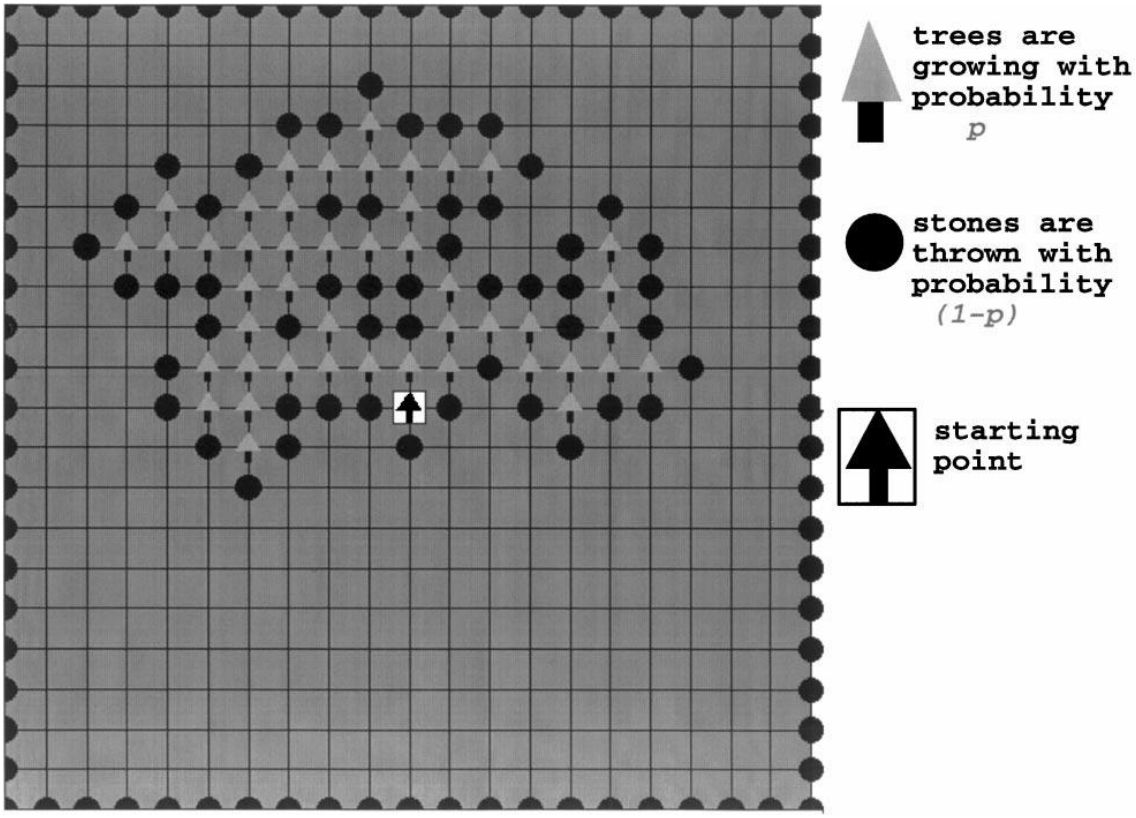

Fig. 2. The cluster of trees generated by the Leath algorithm. The trees are denoted by arrows with the green head, stones denoted by the black circles. The starting point is an arrow with the black head. The forest is generated with the probability $p = 0.5$. Courtesy of E.F. Taylor.

It is known that the average chemical distance $\langle \ell \rangle$ scales as $r^{d_{\min}}$, where various estimates of $d_{\min}$ are $d_{\min} \approx 1.130 \pm 0.005$ [3] and $d_{\min} \approx 1.1307 \pm 0.0004$ [4]. There has been extensive theoretical and computer work done on studying the scaling of $P(\ell|r)$ [5–7]. The complete scaling form of $P(\ell|r)$ which accounts for the finite-size effect, and off-critical behavior has been studied in detail and reported in [8]. Here we would like (i) to present the arguments which underlie the proposed Ansatz and (ii) to describe the algorithm, which we used to generate percolating clusters.

2. Leath algorithm

The Leath algorithm [9–11] is known to be particularly useful for studying structural and physical properties of single percolation clusters. The single clusters are generated in the same way as the forest grow can be imagined (see Fig. 2). First, we start with a single tree, whose seeds are spread to the nearest sites on the square lattice. With some probability p trees grow on the nearest sites. With the probability $1 - p$ the site is occupied by a stone, and thus will be useless for growing. The trees from the surrounding the starting point shell (which we call *chemical shell*) spread their seeds to their nearest neighbors and so on. In each step, a new chemical shell is added. The

process continues until no sites are available for growth or the desired number of shells are generated.

We generate n_d clusters and calculate the number of points $n(\ell, r)$ which are separated from the origin by the geometrical distance r and belong to the certain chemical shell ℓ. We define

$$P'(\ell, r) = \frac{n(\ell, r)}{n_{\text{tot}}}, \tag{1}$$

where n_{tot} is the total amount of points in the n_d clusters.

We also define the probabilities $P'(\ell)$ and $P'(r)$ of a point to have a certain chemical and geometrical distance correspondingly to the starting point

$$P'(\ell) = \frac{\sum_{r=0}^{L} n(\ell, r)}{n_{\text{tot}}} \quad \text{and} \quad P'(r) = \frac{\sum_{\ell=0}^{\ell_{\max}} n(\ell, r)}{n_{\text{tot}}}. \tag{2}$$

Since the Leath algorithm corresponds to the process of selecting a random point on the percolating lattice, those probabilities $P'(\ell, r)$, $P'(\ell)$, and $P'(r)$ are equal to the probabilities that a pair of randomly selected points has certain geometrical and chemical distances, given that they belong to the same and, not necesseraly infinite [1], cluster. The conditional probability $P'(\ell|r) = P'(\ell, r)/P'(r)$ is of special interest here, and has a meaning that two randomly selected points, connected by a percolating path and separated by geometrical distance r, have chemical distance ℓ.

3. The scaling Ansatz

Next, we describe the Ansatz, proposed in [8], for the conditional probability distribution function $P'(\ell|r)$, which includes the effect of the finite size of the system and off-critical behavior. For $\xi > r$,

$$P'(\ell|r) \sim \frac{1}{r^{d_{\min}}} \left(\frac{\ell}{r^{d_{\min}}}\right)^{-g'_\ell} f_1\left(\frac{\ell}{r^{d_{\min}}}\right) f_2\left(\frac{\ell}{L^{d_{\min}}}\right) f_3\left(\frac{\ell}{\xi^{d_{\min}}}\right), \tag{3}$$

where the scaling functions are $f_1(x) = \exp(-ax^{-\phi_1})$, $f_2(x) = \exp(-bx^{\phi_2})$ and $f_3(x) = \exp(-cx)$. Here $\xi \sim |p - p_c|^{-\nu}$ is the correlation length.

The first function f_1 accounts for the lower cut-off due to the constraint $\ell > r$, while f_2 and f_3 account for the upper cut-off due to the finite-size effect and due to the finite-correlation length, respectively (see Fig. 3). Either f_2 and f_3 becomes irrelevant, depending on which of the two values L or ξ is greater. For $L < \xi$, f_2 dominates the upper cut-off, otherwise f_3 dominates. We assume the independence of the finite-size effect and the effect of the concentration of the vacant sites, so that Eq. (3) can be represented as a product of the terms which are responsible for the finite-size effect (f_2) and the effect of the concentration (f_3). Our simulations support this assumption.

[1] We denote by $P(\ell, r)$, $P(\ell) = \int P(\ell, r)\, dr$, and $P(r) = \int P(\ell, r)\, d\ell$ the probabilities that a pair of randomly selected points has certain geometrical and chemical distances, given that they belong to the same *infinite* cluster.

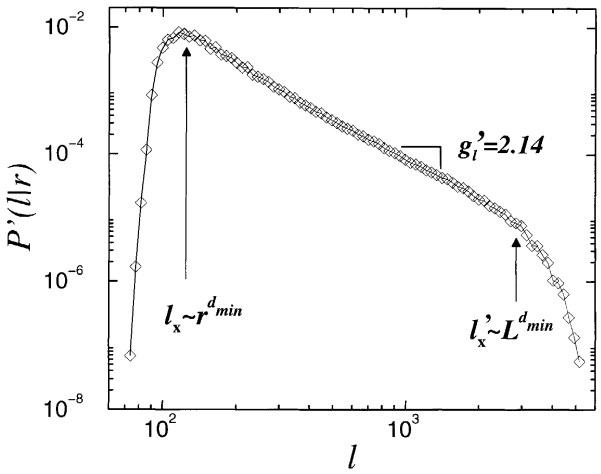

Fig. 3. Log–log plot of $P'(\ell|r)$ for $p = p_c = 0.593$ and for the system size $L = 1024$ and the distance between wells $r = 64$. The straight line regime has slope $g'_\ell = 2.14$. The crossover of $P'(\ell|r)$ at $\ell_x \sim r^{d_{\min}}$ is governed by the function $f_1(x)$ and is due to the constraint $\ell > r$. The crossover of $P'(\ell|r)$ at $\ell'_x \sim L^{d_{\min}}$ is governed by the function $f_2(x)$ and accounts for the finite size of the system.

4. Evidence supporting the scaling Ansatz

4.1. The case of $L \to \infty$ and $p = p_c$

In the case of $L \to \infty$ and $p = p_c$, $P(\ell|r)$ can be obtained from the relation

$$P(\ell|r)P(r) = P(r|\ell)P(\ell) = P(r,\ell)\,, \tag{4}$$

where the conditional probability $P(r|\ell)$ is the probability that two sites, separated by chemical distance ℓ, are a geometric distance r away and belong to the same *infinite* cluster. For isotropic media this probability distribution was studied extensively (see [1,5–7,12,13]). In analogy with the theory of self-avoiding random walks (SAWs) [14], it was proposed [1] that

$$P(r|\ell) = A_l \left(\frac{r}{\ell^{\tilde{\nu}}}\right)^{g_r} f_0\left(\frac{r}{\ell^{\tilde{\nu}}}\right), \tag{5}$$

where $A_\ell \sim 1/\ell^{\tilde{\nu}}$, $\tilde{\nu} = 1/d_{\min} = 0.88 \pm 0.02$, $g_r = 2.2 \pm 0.3$ [15] for $d = 2$ and the scaling function $f_0(x) = \exp(-ax^{\tilde{\delta}})$ with $\tilde{\delta} = (1 - \tilde{\nu})^{-1}$.

The analogy with SAW can be used also to express g_r via other exponents (see [8] for details):

$$g_r = d_f + d_{\min} - 1 \approx 2.04. \tag{6}$$

This value is within the error bars found numerically for g_r in $d = 2$ and $d = 3$ [16]. The scaling Ansatz for $P(\ell|r)$ has been developed in [5–7]. Exactly at $p = p_c$ in the infinite system ($L = \infty$), in analogy to (5),

$$P(\ell|r) = A_r \left(\frac{\ell}{r^{d_{\min}}}\right)^{-g_\ell} f_1\left(\frac{\ell}{r^{d_{\min}}}\right), \tag{7}$$

where $f_1(x) \sim \exp(-ax^{-\phi_1})$ is the scaling function corresponding to f_0 and $A_r \sim 1/r^{d_{\min}}$ is the normalization factor. Comparison of Eqs. (7) and (5) leads to the following relation between exponents: $\phi_1 = 1/(d_{\min}-1)$ and [17] $g_\ell - 1 = (g_r - 1)\tilde{v} + (2 - d_f)\tilde{v}$.

The above relation is true in the case when we restrict the study to the case of the infinite cluster. Using Eq. (6) we find that $g_\ell = 2$ for all $d \geqslant 2$. Note, that the numerical value for $g_\ell \approx 2.04$ found in [8] is very close to this prediction. For the case when two points do not necessarily belong to the *infinite* cluster, it has been shown in [8]

$$P'(\ell|r) = A_r \left(\frac{\ell}{r^{d_{\min}}}\right)^{-g'_\ell} f_1\left(\frac{\ell}{r^{d_{\min}}}\right), \tag{8}$$

where $g'_\ell - g_\ell = (2 - d_f)\tilde{v}$.

The latter relation between exponents g'_ℓ and g_ℓ has simple probabilistic meaning, since a pair of two randomly selected points separated by a chemical distance ℓ should belong to the cluster of chemical size $\ell_0 > \ell$. The probability of this event scales as $\ell^{-\eta\tilde{v}/2}$. Once two points belong to such a cluster, the probability that their chemical distance is equal to ℓ scales the same way as on an infinite cluster and is proportional to ℓ^{-g_ℓ}. Hence the probability that two randomly selected points are separated by a chemical distance ℓ is proportional to the product of these two probabilities $\ell^{-g_\ell - \eta\tilde{v}/2}$, which, by definition, is $\ell^{-g'_\ell}$. Hence $g'_\ell = g_\ell + \eta\tilde{v}/2$.

4.2. The case of finite L and $p = p_c$

In the case of finite L and $p = p_c$, Eq. (3) reduces to

$$P'(\ell|r) \sim \frac{1}{r^{d_{\min}}} \left(\frac{\ell}{r^{d_{\min}}}\right)^{-g'_\ell} f_1\left(\frac{\ell}{r^{d_{\min}}}\right) f_2\left(\frac{\ell}{L^{d_{\min}}}\right), \quad (p = p_c). \tag{9}$$

Numerical tests reveal that $P'(\ell|r)$ has a power-law behavior for $r^{d_{\min}} < \ell < L^{d_{\min}}$ and rapidly vanishes for $\ell < r^{d_{\min}}$ and for $\ell > L^{d_{\min}}$. Fig. 3 illustrates the above observations. Fitting the tails of the distribution by stretched exponentials we find $\phi_1 \approx 7.3$ and $\phi_2 \approx 4.0$.

4.3. The case of $L \to \infty$ and $p \neq p_c$

Finally, in the case of $L \to \infty$ and $p \neq p_c$, the dependence of $P'(\ell|r)$ on p is obtained [8] for very large system size L and for several values of $p \neq p_c$. In this case, the upper cutoff of the distribution Eq. (3) is governed by f_3 and the functional form of the rescaled probability Φ is given by

$$\Phi(\ell/r^{d_{\min}}) \sim f_1\left(\frac{\ell}{r^{d_{\min}}}\right) f_3\left(\frac{\ell}{\xi^{d_{\min}}}\right). \tag{10}$$

For large ℓ, $f_1(x)$ approaches to 1 we are left with an exponential decay of Φ:

$$\Phi(\ell/r^{d_{\min}}) \sim \exp\left(-c\frac{\ell}{\xi^{d_{\min}}}\right). \tag{11}$$

The numerical tests in Ref. [8] confirm this scaling form of $\Phi(\ell/r^{d_{\min}})$.

The scaling form, Eq. (3), is limited to the case when $\xi > r$. For $\xi < r$, the finite-size effects can be neglected, the power-law regime vanishes, and the minimal path can be divided into r/ξ independent blobs each of length $\xi^{d_{\min}}$, so that the distribution $P'(\ell|r)$ approaches Gaussian form with mean $r\xi^{d_{\min}-1}$ and variance $r\xi^{2d_{\min}-1}$.

5. Summary

In summary, we have studied the scaling properties of the shortest paths distribution for fixed two points on the percolating cluster which accounts for the finite-size effect, also off criticality. We propose a plausible scaling hypothesis for the distribution, which is supported by theoretical argument and tested by numerical simulation. The lower and upper cut-offs of the distribution has been numerically observed and fitted successfully by stretched exponential function. Off the critical point the upper cutoff, due to the finite correlation length, becomes a pure exponential form. Finally, we note that when the present approach is extended to study dynamics, new dynamic scaling exponents are found to describe the scaling properties of the distributions of *minimal traveling time, traveling time, and traveling length* of tracer particles [18].

Acknowledgements

We would like to thank Prof. A. Bunde and Prof. D. Stauffer for fruitful discussions. This work is supported by British Petroleum Amoco Exploration Operating Co. Ltd.

References

[1] S. Havlin, D. Ben-Avraham, Adv. Phys. 36 (1987) 695.
[2] P. King, J.S. Andrade, S.V. Buldyrev, N.V. Dokholyan, S. Havlin, Y. Lee, H.E. Stanley, Phys A, 1999, present volume.
[3] H.J. Herrmann, H.E. Stanley, J. Phys. A 21 (1988) L829.
[4] P. Grassberger, J. Phys. A 25 (1992) 5475.
[5] A.U. Neumann, S. Havlin, J. Stat. Phys. 52 (1988) 203.
[6] A. Bunde, H.E. Roman, S. Russ, A. Aharony, A.B. Harris, Phys. Rev. Lett. 69 (1992) 3189.
[7] J.-P. Hovi, A. Aharony, Phys. Rev. E 56 (1997) 172.
[8] N.V. Dokholyan, Y. Lee, S.V. Buldyrev, S. Havlin, H.E. Stanley, P. King, J. Stat. Phys. 93 (1998) 603–613.
[9] P.L. Leath, Phys. Rev. B 14 (1976) 5046.
[10] Z. Alexandrowicz, Phys. Lett. A 80 (1980) 284.
[11] R. Pike, H.E. Stanley, J. Phys. A 14 (1981) L169.
[12] P. Ray, J. Phys. A 18 (1985) L657.
[13] M. Barma, P. Ray, Phys. Rev. B 34 (1986) 3403.
[14] P.-G. de Gennes, Scaling Concepts in Polymer Physics, Ithaca, Cornell University Press, 1979.
[15] M. Porto, A. Bunde, S. Havlin, H.E. Roman, Phys. Rev. E 56 (1997) 1667.
[16] S. Schwarzer, S. Havlin, A. Bunde, Phys. Rev. E, 1998 in press.
[17] A. Bunde, S. Havlin (Eds.), Fractals and Disordered Systems, 2nd ed., Springer, New York, 1996.
[18] Y. Lee, J.S. Andrade, S.V. Buldyrev, N. Dokholyan, S. Havin, P.R. King, G. Paul, H.E. Stanley, Traveling time and traveling length for flow in porous media, Phys. Rev. Lett., submitted.