COMMENTS AND ADDENDA

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Addendum to "Renormalization-group verification of crossover with respect to the lattice anisotropy parameter": Systems with first- and second-neighbor interactions*

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Our recently published renormalization-group treatment of the crossover phenomenon for systems with lattice anisotropy is generalized to isotropic spin systems with competing first- and second-nearest-neighbor interactions, J_1 and J_2. We find that scaling with respect to $R \equiv J_1/J_2$ is valid about $R = 0$; moreover, the crossover exponent is found to be the same as the susceptibility exponent. These results are interesting because the critical-point exponents at $R = 0$ and $R > 0$ should be the same (from universality considerations). Series-expansion analysis confirms this surprising result.

Recently Grover[1] and two of the present authors[2] have shown independently that basic renormalization-group arguments lead directly to the relation $\varphi \equiv \gamma$ for crossover with respect to the anisotropy parameter R in systems with lattice anisotropy.[3] We point out here that similar arguments will lead to the conclusion that there is scaling with respect to the interaction parameter R for an isotropic-spin system with competing first- and second-nearest-neighbor interactions. This result is rather surprising since the exponents do not change from the pure nearest-neighbor values as the critical temperature $T_c(R)$ is approached. As before, our result is valid for any lattice dimension d and isotropically interacting spins of arbitrary dimension n.

Consider, for example, a two-dimensional square lattice with first-and second-nearest-neighbor isotropic-spin interactions. The Hamiltonian is

$$\mathcal{H} = -J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - J_2 \sum_{[ij]} \vec{S}_i \cdot \vec{S}_j , \qquad (1)$$

where $J_1, J_2 > 0$ for ferromagnetic couplings and the first and second sums are over the first- and second-nearest-neighbor spins, respectively. Such a system may be viewed as two interacting layers of two-dimensional square lattices with nearest-neighbor interaction strength J_2 within each layer and $J_1 = RJ_2$ between the two layers (see Fig. 1).

Owing to the general property of translational invariance, it is readily seen that, in general, a

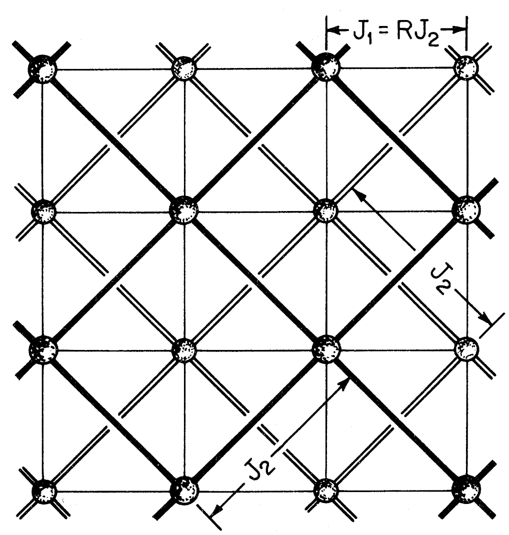

FIG. 1. Two-dimensional square lattice with first- and second-nearest-neighbor interactions viewed as two interacting layers of two-dimensional square lattices with nearest-neighbor interaction strength J_2 within each layer and $J_1 = RJ_2$ between the two layers.

d-dimensional isotropic-spin system with first- and second-nearest-neighbor interactions may be viewed as a finite number (N) of layers of d-dimensional sublattices with nearest-neighbor interaction strength J_2 within the layers and $J_1 = RJ_2$ between the adjacent layers. We can now follow our previous work[2] and apply Wilson's renormalization iterative scheme[4] in each d-dimensional hyperplane and observe (for $R \ll 1$) how the weak-coupling terms between the hyperplanes grow with the iteration procedure. After l- steps of iteration, the effective reduced Hamiltonian has the same form as Eq. (1) of Ref. 2. The difference is that the sums are now over a finite number (N) of layers,

$$H_l = -\sum_{m=1}^{N} \int d^d x [\tfrac{1}{2} \mid \nabla \vec{s}_m(\vec{x}) \mid^2 + Q_l(\vec{s}_m(\vec{x}))] - R_l \sum_{m=1}^{N} \int d^d x \, \vec{s}_m(\vec{x}) \cdot \vec{s}_{m+i}(\vec{x}) , \tag{2}$$

since $\vec{s}_m$ is scaled by a factor $2^{(2-d-\eta)/2}$ in each step of the iteraction, we obtain from Eq. (2)

$$R_l = 2^{(2-\eta)l} R . \tag{3}$$

Here η describes the behavior of the correlation function in the d-dimensional hyperplanes.

During each iteration, the correlation length ξ in each d-dimensional hyperplane is reduced by half. After l iterations we have

$$\xi(\tau_l, R_l) = 2^{-l} \xi(\tau, R) , \tag{4a}$$

where $\tau \equiv T_c(0)/[T - T_c(0)]$ is the reduced temperature. Similarly, we can show that[1] the Gibbs potential G scales as

$$G(\tau_l, R_l) = 2^{dl} G(\tau, R) . \tag{4b}$$

The usual renormalization-group argument[4] leads to the scaling relation

$$\tau_l = 2^{l/\nu} \tau , \tag{5}$$

where ν is the exponent for the correlation length associated with each d-dimensional hyperplane. Therefore we prove from Eqs. (3)–(5) the generalized homogeneous scaling laws

$$\xi(\lambda^{1/\nu}\tau, \lambda^{2-\eta} R) = \lambda^{-1} \xi(\tau, R) , \tag{6a}$$

$$G(\lambda^{1/\nu}\tau, \lambda^{2-\eta} R) = \lambda^d G(\tau, R) . \tag{6b}$$

Thus we are able to prove that there is scaling with respect to the interaction parameter R. This is true in spite of the fact that the critical exponents do not change as the critical temperature $T_c(R)$ is approached.

Equation (6b) may be put into the more familiar form

$$G(\tau, R) = \tau^{2-\alpha} F(R/\tau^{\varphi}) , \tag{7}$$

where $\alpha = 2 - d\nu$ is the specific-heat exponent of the d-dimensional hyperplanes and $\varphi = (2-\eta)\nu$. Thus we verify that the crossover exponent φ is the same as the susceptibility exponent γ.

We emphasize that although ν, η, α, and γ are the critical exponents of the d-dimensional hyperplanes, from universality considerations they are also the true exponents of the original system. The critical temperature $T_c(R)$, however, is expected to change with the interaction parameter R.

Our result easily generalizes to the situation where an external magnetic field $\vec{h}$ is included. Since the additional term $\vec{h} \cdot \vec{s}$ in the reduced Hamiltonian is uncoupled from the renormalization scheme and $\vec{s}$ is scaled by a factor of $2^{(2-d-\eta)/2}$ in each step of the iteration, we must have $h_l = 2^{(2+d-\eta)l/2} h$. Therefore Eq. (6b) becomes

$$G(\lambda^{1+d/2-\eta/2} h, \lambda^{1/\nu}\tau, \lambda^{2-\eta} R) = \lambda^d G(h, \tau, R) . \tag{8}$$

Differentiating Eq. (8) successively, we obtain

$$\left(\frac{\partial^k \chi}{\partial R^k}\right)_{R=0} \sim \tau^{-\gamma_k} , \tag{9}$$

where $\chi \equiv (\partial^2 G/\partial h^2)_{\tau, R}$ is the magnetic susceptibility and $\gamma_k = (k+1)\gamma$. We have verified this rather surprising result numerically for the sc, fcc, bcc, and diamond lattices for the Ising, planar, and Heisenberg spin systems (see Table I).[5] Elsewhere,[6] we have generalized this approach to systems with long-range and dipolar interactions.

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TABLE I. Padé approximants of γ_k based on the high-temperature series for χ on the fcc lattice with competing first- and second-nearest-neighbor ferromagnetic couplings (··· indicates that the element is complex).

Padé element (D, N)	γ_1	γ_2	γ_3	γ_4	γ_5
Ising ($n=1$)—tenth-order series					
(2, 2)	2.4714	3.7541	5.0628	6.3842	7.7085
(2, 3)	2.4974	3.7340	4.8332	6.5129	
(3, 2)	2.4851	3.7246	5.1534	6.4263	
(3, 3)	2.4769	3.7479	4.9958		
(3, 4)	2.5254	3.7689			
(4, 3)	2.5025	3.7612			
(4, 4)	2.5110				
Planar ($n=2$)—ninth-order series					
(2, 2)	2.5632	3.8895	5.2377	6.5893	
(2, 3)	2.6133	3.9044	5.1260		
(3, 2)	2.5546	3.8955	5.2802		
(3, 3)	···	3.8857			
(3, 4)	2.6697				
(4, 3)	2.5830				
Heisenberg ($n=3$)—ninth-order series					
(2, 2)	2.6238	3.9797	5.3555	6.7315	
(2, 3)	2.6933	4.0223	5.3028		
(3, 2)	2.5823	3.9625	5.3562		
(3, 3)	···	3.9536			
(3, 4)	2.7827				
(4, 3)	2.6054				

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[1]M. K. Grover, Phys. Lett. A 44, 253 (1973). Here a spin-independent term is neglected.
[2]T. S. Chang and H. E. Stanley, Phys. Rev. B 8, 4435 (1973).
[3]L. L. Liu and H. E. Stanley, Phys. Rev. Lett. 29, 927 (1972); Phys. Rev. B 8, 2279 (1973). Here the result $\gamma_k=(k+1)$, γ was proved rigorously for the Ising case ($n=1$) for $k=1$, 2, 3 and for all n for $k=1$. Since the proof is valid when the number of layers is finite, it provides a valuable check on the Padé-approximant analysis as well as on the renormalization-group theory. Note that in previous work, we have defined $R'\equiv J_2/J_1$.
[4]K. G. Wilson, Phys. Rev. B 4, 3174, (1971); 4, 3184 (1971); K. G. Wilson and J. Kogut, Phys. Rep. C 12, 75 (1974), and references therein.
[5]Only one typical set of Padé results is given in Table I. Other results are very similar and have been documented along with the series in National Auxiliary Publication Service (NAPS). See NAPS document No. 02496 for 120 pages of supplementary material. Order from ASIS/NAPS c/o Microfiche Publications, 440 Park Avenue South, New York, N.Y. 10016. Remit in advance for each NAPS accession number. Make checks payable to Microfiche Publications. Photocopies are \$18.50. Microfiche are \$1.50. Outside of the United States and Canada, postage is \$2.00 for a photocopy or \$.50 for a fiche. The first 15 pages of NAPS document No. 02496 contains Padé approximants to high temperature series on the fcc, bcc, sc, and diamond lattices, followed by 120 pages containing the series for $\overline{\chi}$, $\overline{\chi}_{\text{staggered}}$, μ_2, and $\overline{C}_H$, for the Hamiltonian (1) on the fcc, bcc, and sc lattices. NAPS document No. 02245 contains series for $\overline{\chi}$, μ_2, and $\overline{C}_H$ for (1) on the diamond lattice. The series for (1) include spin dimensionalities $n=1$, 2, and 3.
[6]T. S. Chang, L. L. Liu, and H. E. Stanley (report of work prior to publication).