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Scaling behavior in economics: empirical results and modeling of company growth

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1. – Introduction

Statistical physics has undergone many changes in emphasis during the last few decades. The seminal works of the '60s and '70s on critical phenomena [1] and of the '80s and '90s on fractal geometry [2] provided physicists with a new set of tools to study nature [3, 4]. Fields such as biophysics, medicine, geomorphology, geology, evolution, ecology or meteorology are now common areas of application of statistical physics.

In particular, several research groups have turned their attention to problems in economics [5, 6] and finance [7-13]. In this article, we extend the study of ref. [6] on the growth rate of manufacturing companies. One motivation for the present study is the

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considerable recent interest in economics in developing a richer theory of the firm [14-32]. In standard microeconomic theory, a firm is viewed as a production function for transforming inputs such as labor, capital, and materials into output [16, 21, 27]. When dynamics are incorporated into the model, the source of the link between production in one period and production in another arises because of investment in durable, physical capital and because of technological change (which in turn can arise from investments in research and development). Recent work on firm dynamics emphasizes the effect of how firms learn over time about their efficiency relative to competitors [20, 33, 34]. The production dynamics captured in these models are not, however, the only source of actual firm dynamics. Most notably, the existing models do not account for the time needed to assemble the organizational infrastructure needed to support the scale of production that typifies modern corporations.

We studied all United States (US) manufacturing publicly-traded firms from 1974 to 1993. The source of our data is Compustat which is a database on all publicly-traded firms in the US. Compustat obtains this information from reports that publicly traded companies must file with the US Securities and Exchange Commission. The database contains a large amount of information on each company. Among the items included are "sales," "cost of goods sold," "assets," "number of employees," and "property, plant & equipment."

Another item provided for each company is the Standard Industrial Classification (SIC) code. In principle, two companies in the same primary SIC code are in the same market; that is, they compete with each other. In practice, defining markets is extremely difficult [35]. More important for our analysis, virtually all modern firms sell in more than one market. Companies that operate in different markets do report some disaggregated data on the different activities. For example, while Philip Morris was originally a tobacco producer, it is also a major seller of food products (since its acquisition of General Foods) and of beer (since its acquisition of Miller Beer). Philip Morris does report its sales of tobacco products, food products, and beer separately. However, companies have considerable discretion in how to report information on their different activities, and differences in their choices make it difficult to compare the data across companies.

In this paper, the only use we make of the primary SIC codes in Compustat is to restrict our attention to manufacturing firms. Specifically, we include in our sample all firms with a major SIC code from 2000–3999. We do not use the data from the individual business segments of a firm, nor do we divide up the sample according to SIC codes. We should acknowledge that this choice is at odds with the mainstream of economic analysis. In economics, what is commonly called the "theory of the firm" is actually a theory of a business unit. To build on the Philip Morris example, economists would likely not use a single model to predict the behavior of Philip Morris. At the very least, they would use one model for the tobacco division, one for the food division, and one for the beer division. Indeed, given the available data, they might construct a completely separate model of, say, the sales of Maxwell House coffee. Absent any effect of the output of one of Philip Morris' products on either the demand for or costs of its other products, the models of the different components of the firm would be completely separate. Because the standard model of the firm applies to business units, it does not yield any prediction about the distribution of the size of actual, multi-divisional firms or their growth rates.

On the other hand, the approach we take in this study is part of a distinguished tradition. First, there is a large body of work by Economics Nobel laureate H. Simon [22] and various co-authors that explored the stochastic properties of the dynamics of firm growth. Also, in a widely cited article (that nonetheless has not had much impact on mainstream economic analysis), R. Lucas, also a Nobel laureate, suggests that the distribution of firm size depends on the distribution of managerial ability in the economy rather than on the factors that determine size in the conventional theory of the firm [23].

In summary, the objective of our study is to uncover empirical scaling regularities about the growth of firms that could serve as a test of models for the growth of firms. We find: i) the distribution of the logarithm of the growth rates for firms with approximately the same size displays an exponential "tent-shaped" form, and ii) the fluctuations in the growth rates—measured by the width of this distribution—scale as a power law with firm size. The width of the distribution has a tendency to grow with time, but the shape of the distribution remains tent-shaped.

The paper is organized as follows: in sect. 2, we review the economics literature on the growth of companies. In sect. 3, we present our empirical results for publicly-traded US manufacturing companies. In sect. 4, we propose and discuss models that shed some light on those results. Finally, in sect. 5, we present concluding remarks and discuss questions raised by our results.

2. – Background

In 1931, the French economist Gibrat proposed a simple model to explain the empirically observed size distribution of companies [14]. He made the following assumptions: i) the growth rate R of a company is independent of its size (this assumption is usually referred to by economists as the *law of proportionate effect*), ii) the successive growth rates of a company are uncorrelated in time, and iii) the companies do not interact.

In mathematical form, Gibrat's model is expressed by the stochastic process:

$$S_{t+\Delta t} = S_t(1 + \varepsilon_t), \tag{1}$$

where $S_{t+\Delta t}$ and S_t are, respectively, the size of the company at times $(t + \Delta t)$ and t, and ε_t is an uncorrelated random number with some bounded distribution and variance much smaller than one (usually assumed to be Gaussian). Hence $\log S_t$ follows a simple random walk and, for sufficiently large time intervals $T \gg \Delta t$, the growth rates

$$R_T \equiv \frac{S_{t+T}}{S_t} \tag{2}$$

are log-normally distributed. If we assume that all companies are born at approximately the same time and have approximately the same initial size, then the distribution of company sizes is also log-normal.

A considerable advantage of Gibrat's model is that it yields testable hypotheses. The law of proportionate effect implies that the mean growth rate and the fluctuations of the growth rate are independent of size. In fact, however, the fluctuations of the growth rate measured by the standard deviation $\sigma(S)$ decline with an increase in firm size. This was first observed by Singh and Whittington [36] and confirmed by others [6, 37-41]. The negative relationship between growth fluctuations and size is not surprising because large firms are likely to be more diversified. Singh and Whittington state that the decline of the standard deviation with size is not as rapid as if the firms consisted of independently operating subsidiary divisions. The latter would imply that the relative standard deviation decays as $\sigma(S) \sim S^{-1/2}$ [36]. This confirms the common-sense view that the performance of different parts of a firm are related to each other.

The situation for the mean growth rate is less clear. Singh and Whittington [36] consider the assets of firms and observe that the mean growth rate increases slightly with size. However, the work of Evans [37] and Hall [38], using the number of employees to define the company's size, suggests that the mean growth rate declines slightly with size. Dunne *et al.* [39] emphasize the effect of the failure rate of firms and the effect of the ownership status (single- or multi-unit firms) on the relation between size and mean growth rate. They conclude that the mean growth rate is always negatively related with size for single-unit firms; but for multi-unit firms, the growth rate increases modestly with size because the reduction in their failure rates overwhelms a reduction in the growth of nonfailing firms [39].

Another testable implication of Gibrat's law is that the growth rate of a firm is uncorrelated in time. However, the empirical results in the literature are not conclusive. Singh and Whittington [36] observe positive first-order correlations in the 1-year growth rate of a company (persistence of growth) whereas Hall [38] finds no such correlations. The possibility of negative correlations (regression towards the mean) has also been suggested [42, 43].

3. – Empirical results

In this section, we study the distribution of company sizes and growth rates. To do so, one problem that must be confronted is the definition of firm size. If all companies produced the same good (steel, say), then we could use a physical measure of output, such as tons. We are, however, studying companies that produce different goods for which there is no common physical measure of output. An obvious solution to the problem is to use the dollar value of output: the sales. A general alternative to measuring the size of output is to measure input. Again, since companies produce different goods, they use different inputs. However, virtually all companies have employees. As a result, some economists have used the number of employees as a measure of firm size. Three other possibilities involve the dollar value of inputs, such as the "cost of goods sold," "property, plant & equipment," or "assets." As we discuss below, we obtain similar results for all of these measures. We begin by describing the growth rate of sales. To make the values of sales in different years comparable, we adjust all values to 1987 dollars by the GNP price deflator.

Since the law of proportionate effects implies a multiplicative process for the growth of companies, it is natural and more convenient to study the logarithm of sales. We thus define

$$s_0 \equiv \ln S_0 \tag{3}$$

and the corresponding growth rate

$$r_1 \equiv \ln R_1 = \ln \frac{S_1}{S_0}\,, \tag{4}$$

where S_0 is the size of a company in a given year and S_1 its size the following year.

3·1. *Size distribution of publicly-traded companies.* – Stanley *et al.* determined the size distribution of publicly-traded manufacturing companies in the US [44]. They

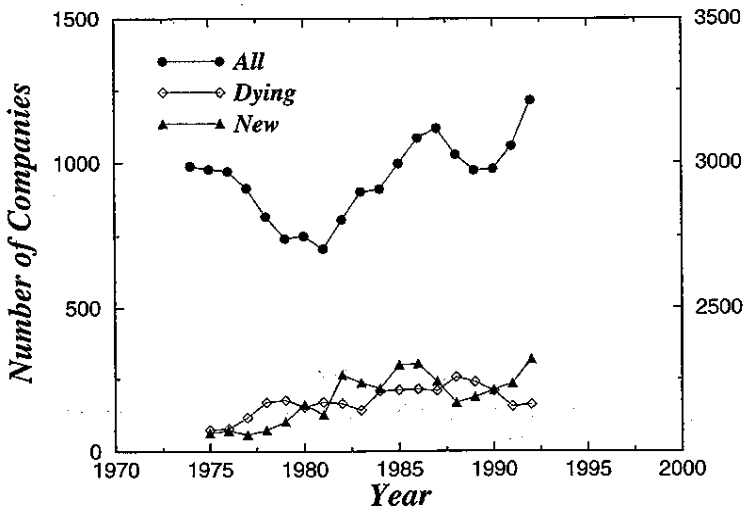

Fig. 1. – Number of publicly-traded manufacturing companies in the US for the period 1974–1993 (right scale). Also shown is the number of companies entering the market and the number of companies leaving (left scale).

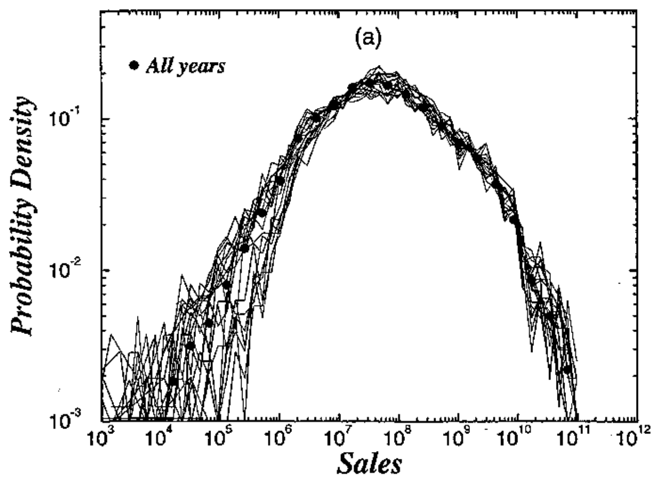

Fig. 2a. – Probability density of the logarithm of the sales for publicly-traded manufacturing companies (with standard industrial classification index of 2000-3999) in the US for *each* of the years in the 1974–1993 period. All the values for sales were adjusted to 1987 dollars by the GNP price deflator. Also shown (solid circles) is the average over the 20 years. It is visually apparent that the distribution is approximately stable over the period.

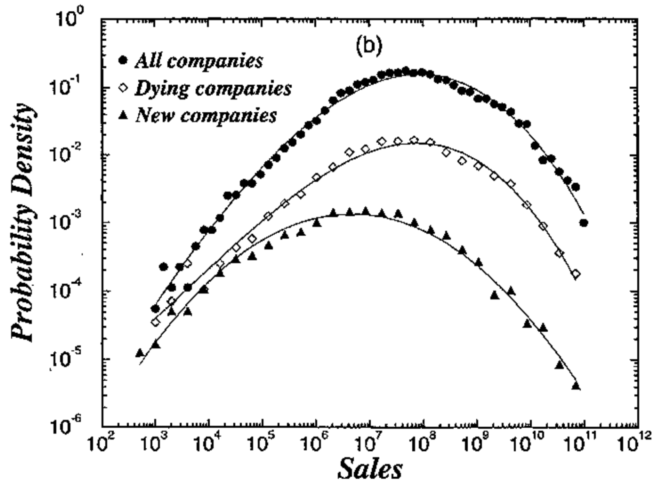

Fig. 2b. – Probability density of the logarithm of sales for all the manufacturing companies, for the companies entering the market (shifted by a factor of 1/10), and for the companies leaving the market (shifted by a factor of 1/100), averaged over the 1974–1993 period. The distribution of new companies can be described to first approximation by a log-normal while the other distributions are better fitted by the exponential of a third-order polynomial. Notice that the distributions of all companies and of dying companies are nearly identical. This suggests a nearly constant dependence of the dying probability on size.

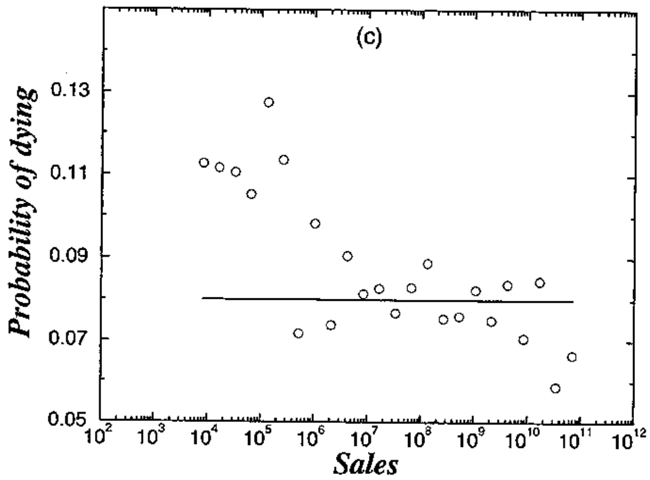

Fig. 2c. – Plot of the fraction of "dying" companies by size. We define this probability as the ratio of the number of companies of a given size dying every year over the total number of companies of that size. The horizontal straight line is a guide for the eye for companies with sales above 10^6.

found that for 1993 the data fit to a good degree of approximation a log-normal distribution. These results have been recently confirmed by Hart and Oulton [45] for a sample of approximately 80000 United Kingdom companies. Here, we present a study of the distribution for a period of 20 years (from 1974 to 1993).

Figure 1 shows the total number of publicly-traded manufacturing companies present in the database each year. We also plot the number of *new* companies and of "*dying*" companies (*i.e.*, companies that leave the database because of merger, change of name or bankruptcy).

Figure 2(a) shows the distribution of firm size in each year from 1974–1993. Particularly above the lower tails, the distributions lie virtually on top of each other. Thus the distribution is stable over this period. This is a surprising result, when we compare it with the predictions of the Gibrat model. Equation (1) implies that the distribution of sizes of companies should get broader with time. In fact, the variance of the distribution should increase linearly in time. Thus, we must conclude that other factors, not included in Gibrat's assumptions, must have important roles.

One obvious factor not captured by the Gibrat assumption is the entry of new companies. Figure 2(b) shows that the size distribution of new publicly-traded companies is approximately a log-normal with an average value slightly smaller than the average of all companies. A possible explanation for this result is that some of these new companies are the result of mergers or of the breaking up of existing large companies.

Another factor not included in Gibrat's assumptions is the "dying" of companies. As shown in figs. 2(b), 2(c), this distribution is quite similar to the distribution for all companies. Thus, it suggests that the probability for a company to leave the market, whether by merger, change of name, or bankruptcy, is nearly independent of size.

When analyzing the data, it is important to consider the high level of the noise in the tails. The use of equally spaced bins in building a histogram from the data is the most straightforward method. However, it implies very noisy results for the tails because of the small number of data points in that region. One way to solve this problem, especially if some knowledge of the shape of the distribution exists, is to take bins chosen with such lengths that all of them receive approximately the same number of data points. In fact, we used equally spaced bins on a *logarithmic scale*, *i.e.* all firms with sales values falling into an interval between 8^k and 8^{k+1} with k an integer belong to one bin.

3·2. *The distribution of annual growth rates.* – The distribution $p(r_1\,|\,s_0)$ of the growth rates from 1974 to 1993 is shown in fig. 3 for three different values of the initial sales [46]. Remarkably, these curves display a simple "tent-shaped" form. Hence the distribution is not Gaussian—as expected from the Gibrat approach [14]— but rather is exponential [6],

$$p(r_1\,|\,s_0) = \frac{1}{\sqrt{2}\,\sigma_1(s_0)}\exp\left[-\frac{\sqrt{2}\,|r_1-\overline{r}_1(s_0)|}{\sigma_1(s_0)}\right]. \tag{5}$$

The straight lines shown in fig. 3 are calculated from the average growth rate $\overline{r}_1(s_0)$ and the standard deviation $\sigma_1(s_0)$ obtained by fitting the data to eq. (5). An implication of this result is that the distribution of the growth rate has much broader tails than would be expected for a Gaussian distribution.

3·3. *Mean growth rate.* – Economists have studied the relationship between mean growth rate and firm size. Typically they do so by running a regression of growth rates on firm size sometimes with other control variables included. Rather than use a regression function, we undertake a graphical analysis of the mean growth rate. Figure 4(a) displays $\overline{r}(s_0)$ as a function of initial size S_0 for several years. Although the data are quite noisy, they suggest that there is *no* significant dependence of the mean growth rate on S_0. Least-square fits of the individual curves to a form $\overline{r}(s_0)\sim\ln S_0$ lead to estimates of the proportionality constant which are very small in magnitude ($<10^{-2}$), and whose sign can be positive or negative depending on the year. Our analysis suggests that if a dependence exists, it is *very* weak for any range of sizes.

The analysis for the average of the nineteen 1-year periods, which is displayed in fig. 4(b), confirms this observation. Furthermore, the figure suggests that the results do not change when we consider other definitions of the size of a company.

3·4. *Standard deviation of the growth rate.* – Next, we study the dependence of $\sigma_1(s_0)$ on s_0. As is apparent from fig. 3, the width of the distribution of growth rates decreases with increasing s_0. We find that $\sigma_1(s_0)$ is well approximated for 8 orders of magnitude (from sales of less than 10^3 dollars up to sales of more than 10^{11} dollars) by

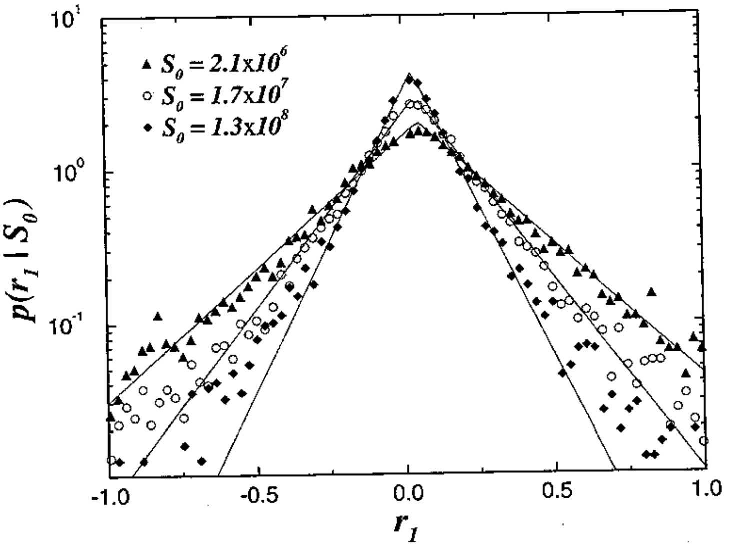

Fig. 3. – Probability density $p(r_1 \mid s_0)$ of the growth rate $r \equiv \ln(S_1/S_0)$ for all publicly-traded US manufacturing firms in the 1994 Compustat database with Standard Industrial Classification index of 2000–3999. The distribution represents all annual growth rates observed in the 19-year period 1974–1993. We show the data for three different bins of initial sales (with sizes increasing by powers of 8): $8^7 < S_0 < 8^8$, $8^8 < S_0 < 8^9$, and $8^9 < S_0 < 8^{10}$. The solid lines are exponential fits to the empirical data close to the peak. We can see that the wings are somewhat "fatter" than what is predicted by an exponential dependence.

the law [6]

$$\sigma_1(s_0) \sim \exp[-\beta s_0], \tag{6}$$

where $\beta = 0.20 \pm 0.03$. Equation (6) implies the scaling law

$$\sigma_1(S_0) \sim S_0^{-\beta}. \tag{7}$$

Figure 4(c) displays σ_1 *vs.* S_0, and we can see that eq. (7) is indeed verified by the data.

Also of interest is the width of the distribution of final sizes $S_1 \equiv S_0 \exp[r_1]$, that we designate by $\Sigma_1(S_0)$. We can express Σ_1 as

$$\Sigma_1(S_0)^2 = \langle S_1^2 \rangle - \langle S_1 \rangle^2. \tag{8}$$

Taking $\overline{r_1}(s_0) \approx 0$, simple integrations lead to the results

$$\langle S_1 \rangle = \int_{-\infty}^{+\infty} S_1 p(r_1 \mid S_0)\, \mathrm{d}r_1 = \frac{S_0}{1 - \sigma_1^2/2} \tag{9}$$

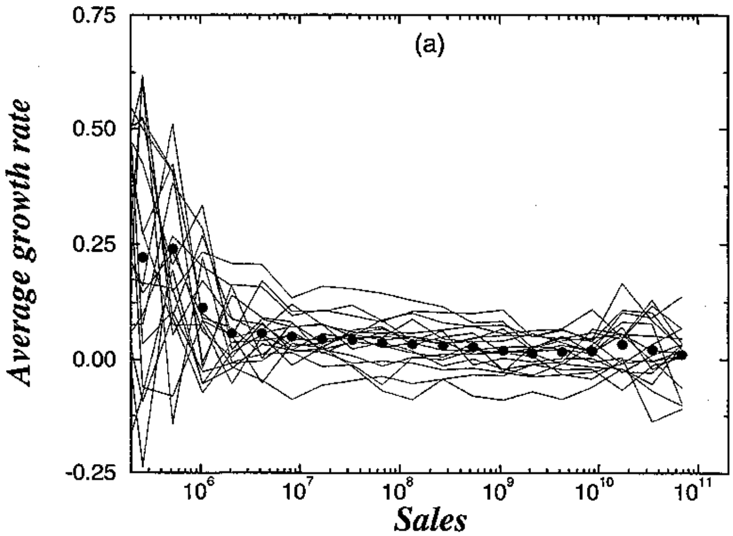

Fig. 4a. – Mean 1-year growth rate $\bar{r}_1(s_0)$ for several years. It is visually apparent that the data are quite noisy, and that there is no significant dependence on S_0 (at most a logarithmic dependence with a very small coefficient). Also displayed is the mean growth rate for the 18-year period in Compustat.

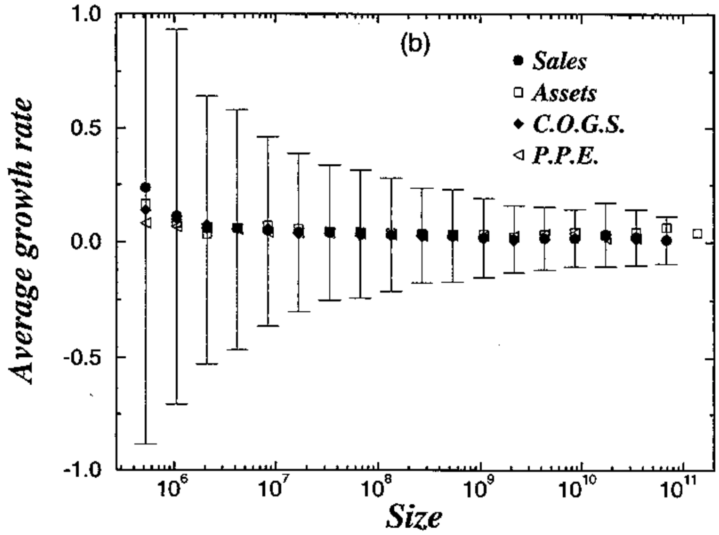

Fig. 4b. – Average for the 19 years of $\bar{r}_1(s_0)$ for several size definitions: sales, assets, cost of goods sold and plant property and equipment. Error bars corresponding to one standard deviation are shown for sales—values for the other quantities are nearly identical. Again, no significant dependence on S_0 is found. Although it seems likely that the slightly positive value of $\bar{r}(s_0)$ is a real effect, we cannot rule out the possibility of a bias of the data towards successful companies.

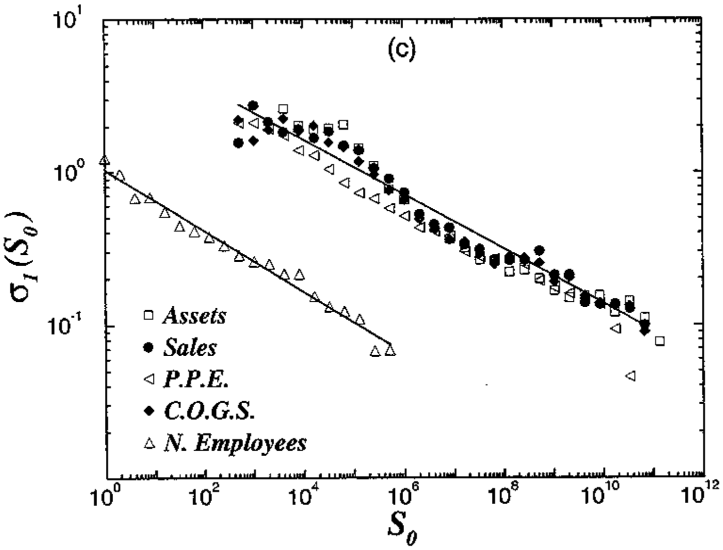

Fig. 4c. – Standard deviation of the 1-year growth rates for different definitions of the size of a company as a function of the initial values. Least-squares power law fits were made for all quantities leading to the estimates of β: 0.18 ± 0.03 for "assets," 0.20 ± 0.03 for "sales," 0.18 ± 0.03 for "number of employees," 0.18 ± 0.03 for "cost of goods sold," and 0.20 ± 0.03 for "plant, property and equipment." The straight lines are guides for the eye and have slopes 0.19.

and

$$\langle S_1^2 \rangle = \int_{-\infty}^{+\infty} S_1^2 p(r_1 \,|\, S_0)\, \mathrm{d}r_1 = \frac{S_0^2}{1 - 2\sigma_1^2}\,. \tag{10}$$

Replacing these results onto (8) and expanding in Taylor series, we obtain

$$\Sigma_1(S_0)^2 = S_0^2\big(1 + 2\sigma_1^2 + 4\sigma_1^4 + \ldots - 1 - \sigma_1^2 - 3\sigma_1^4/4 + \ldots\big) \approx (S_0\sigma_1)^2(1 + 13\sigma_1^2/4)\,. \tag{11}$$

Thus, to first order, we obtain

$$\Sigma_1(S_0) \sim S_0^{1-\beta}\,. \tag{12}$$

Note that the integral (10) converges only if $\sigma_1 < 1/\sqrt{2}$, which holds for companies with sales larger than 10^6 dollars. For smaller companies, the values of Σ_1 computed directly from the Compustat data fluctuate dramatically from one year to the next, as one would expect for the random variable with infinite variance.

3·5. *Other measures of size.* – In order to test further the robustness of our findings, we perform a parallel analysis for the number of employees. We find that the analogs of $p(r_1 \,|\, s_0)$ and $\sigma_1(s_0)$ behave similarly. For example, fig. 4(c) shows the standard deviation of the number of employees, and we see that the data are linear over roughly

5 orders of magnitude, from firms with less than 10 employees to firms with almost 10^6 employees. The slope $\beta = 0.18 \pm 0.03$ is the same, within the error bars, as found for the sales.

As shown in fig. 4(c), we find that eqs. (5) and (7) accurately describe three additional indicators of a company's size, i) assets (with exponent $\beta = 0.18 \pm 0.03$), ii) cost of goods sold ($\beta = 0.18 \pm 0.03$), and iii) property, plant & equipment ($\beta = 0.20 \pm 0.03$).

3·6. *The T-year growth rates.* – Another relevant question is the validity of eq. (5) for larger periods of time, *i.e.*, if we consider the T-year growth rate r_T, will we get a similar distribution or not? The analysis of the data shows that the distribution of growth rates for T as large as 8 years does not follow a log-normal distribution.

We find that for $T \leqslant 8$ the distribution of growth rates approximately follows an exponential distribution; cf. fig. 5(a). For $T = 16$ the results are not clear due to the noise.

Finally, we study the dependence of the width of the distribution, for a given value of s_0, on time. Figure 5(b) suggests that $\sigma_T(s_0)$ grows as a logarithm or a small power of T.

For large company sizes the growth of σ_T can to some degree be approximated by $\sqrt{T}$, which is expected for independent successive annual growth rates. However for small companies, σ_T grows more slowly than $\sqrt{T}$, thus suggesting that one-year

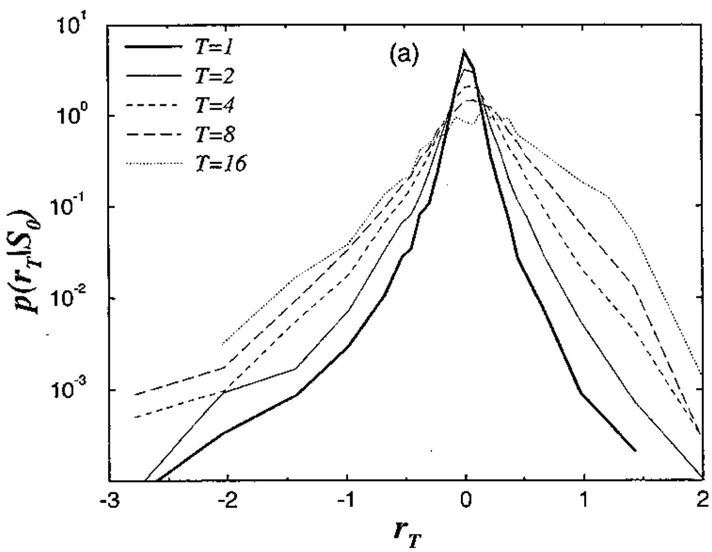

Fig. 5a. – Probability density of the T-year growth rate for companies with initial size of $8^8 < S_0 < 8^9$. It is visually apparent that, at least for $T \leqslant 8$, the distribution is well approximated in its central part by eq. (5).

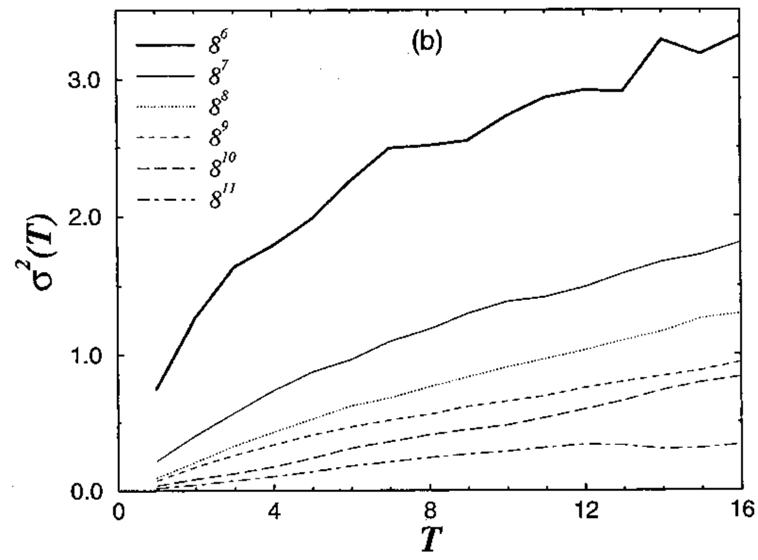

Fig. 5b. – Plot of the average square width of the distribution σ_T^2 as a function of T for different values of S_0. It is clear that σ^2 increases slower than linearly. This result implies anti-correlations in the successive one-year growth rates.

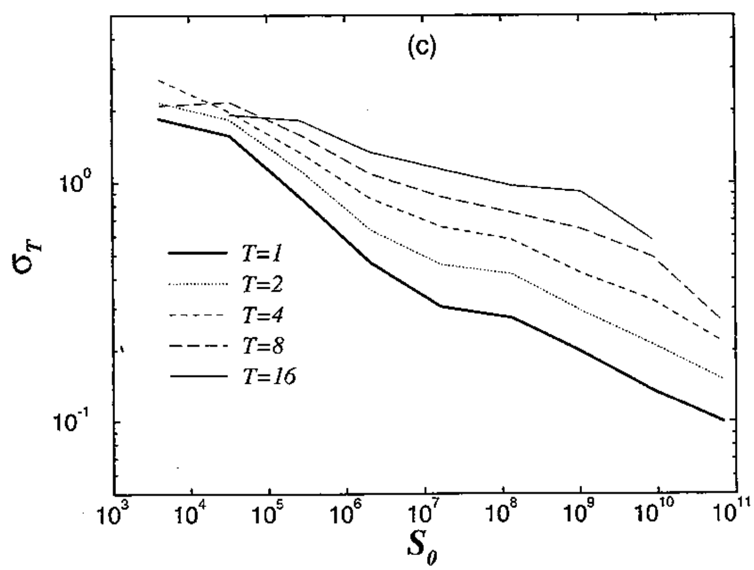

Fig. 5c. – Plot of the average width of the distribution σ_T, as a function of S_0 for different values of T. It is clear that the size dependence of σ_T becomes weaker for larger values of $S_0\,T$.

growth rates are anticorrelated. Our data suggest also that the exponent β is not universal but decreases with T (see fig. 5(c)).

3·7. *Discussion.* – What is remarkable about eqs. (5) and (7) is that they approximate the growth rates of a diverse set of firms. They differ not only in their size but also in what they manufacture. The conventional economic theory of the firm is based on production technology, which varies from product to product. Conventional theory does not suggest that the processes governing the growth rate of car companies should be the same as those governing, *e.g.*, pharmaceutical or paper firms. Indeed, our findings are reminiscent of the concept of universality found in statistical physics, where different systems can be characterized by the same fundamental laws, independent of "microscopic" details. Thus, we can pose the question of the universality of our results: is the measured value of the exponent β due to some averaging over the different industries, or is it due to a universal behavior valid across all industries? As a "robustness check," we split the entire sample into two distinct intervals of SIC codes. It is visually apparent in fig. 6(a) that the same behavior holds for the different industries. Thus, we can conclude that our results are indeed universal across different manufacturing industries in the US.

In statistical physics, scaling phenomena of the sort that we have uncovered in the sales and employee distribution functions are sometimes represented graphically by plotting a suitably "scaled" dependent variable as a function of a suitably "scaled"

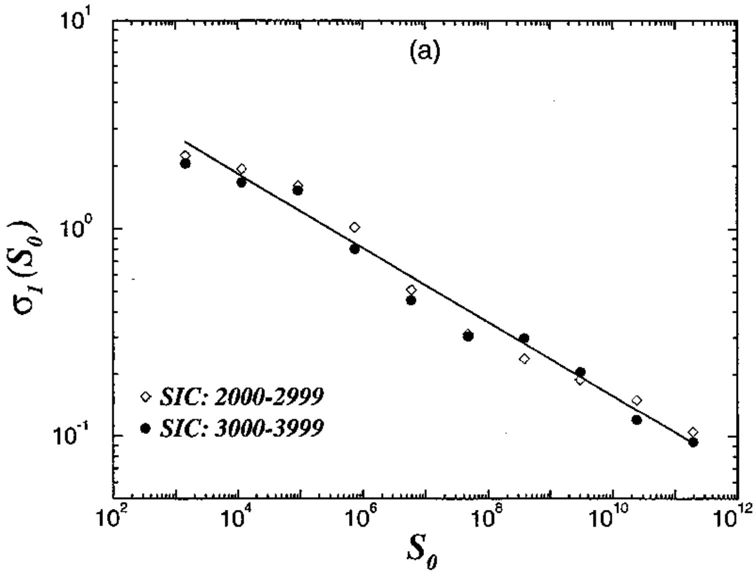

Fig. 6a. – Dependence of σ_1 on S_0 for two subsets of the data corresponding to different values of the SIC codes. In principle, companies in different subsets operate in different markets. The figure suggests that our results are universal across markets.

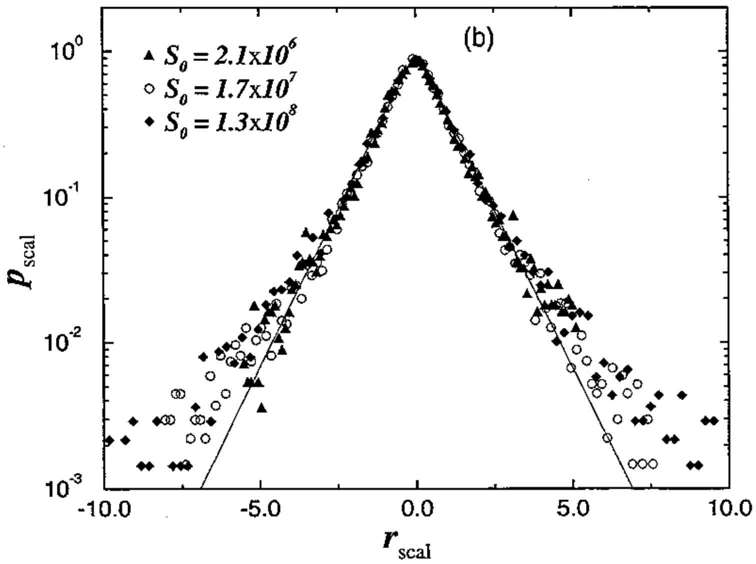

Fig. 6b. – Scaled probability density $p_{\text{scal}} \equiv \sqrt{2}\,\sigma_1(s_0)\,p(r_1\,|\,s_0)$ as a function of the scaled growth rate $r_{\text{scal}} \equiv \sqrt{2}\,[r_1 - \bar{r}_1(s_0)]/\sigma_1(s_0)$. The values were rescaled using the measured values of $\bar{r}_1(s_0)$ and $\sigma_1(s_0)$. All the data collapse upon the universal curve $p_{\text{scal}} = \exp[-|r_{\text{scal}}|]$ as predicted by eqs. (5) and (6).

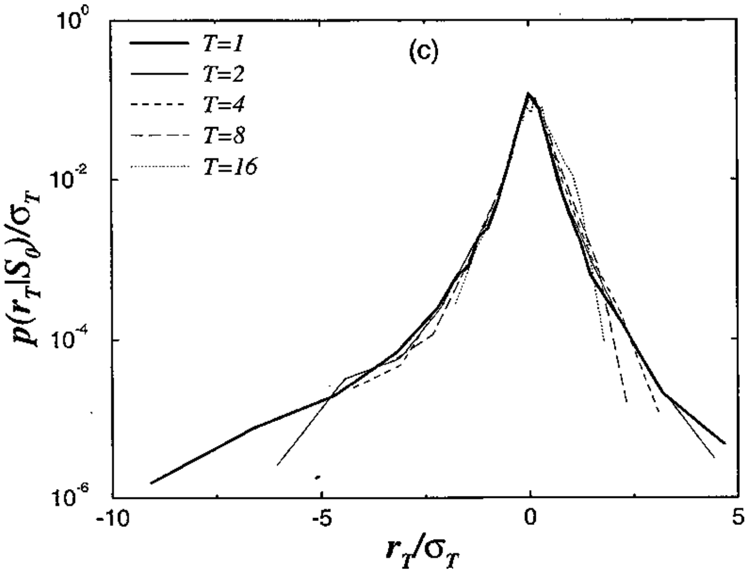

Fig. 6c. – Similar scaling plot for the data from fig. 5a. Again, we can see that all the data collapse onto a single curve.

independent variable. If scaling holds, then the data for a wide range of parameter values are said to "collapse" upon a *single* curve. To test the present data for such data collapse, we plot in fig. 6(b) the scaled probability density $p_{\rm scal} \equiv \sqrt{2}\,\sigma(s_0)\,p(r_1\,|\,s_0)$ as a function of the scaled growth rates of both sales and employees $r_{\rm scal} \equiv \sqrt{2}\,[r_1 - r_1(s_0)]/\sigma(s_0)$. The data collapse upon the single straight line $p_{\rm scal} = \exp[-|r_{\rm scal}|]$ shows small but consistent deviations for large growth rates from the exponential distribution in eq. (5). Thus eq. (5) can be regarded only as a first-order approximation to reality. Our results for i) cost of goods sold, ii) assets, and iii) property, plant & equipment are equally consistent with such scaling. Figure 6(c) represents the analogous plot for growth rates for different time periods T. It can be seen that the shape of the distribution remains practically unchanged for larger periods of time $T > 1$. Regardless of the exact validity of eqs. (5) and (7), it is remarkable that the shape of the distribution is similar for all company sizes and does not converge to a Gaussian, even for large T—as the Gibrat model (eq. (1)) would predict.

The high degree of similarity in the behavior of sales, the number of employees, and of the other measures of size that we studied points to the existence of large correlations among those quantities, as one would expect.

4. – Stochastic modeling

In this section we will present and discuss models that, although very simple, may give some insight into the empirical results. First, we look into the problem of the distribution of growth rates. The generally weak assumptions underlying the central limit theorem suggest that the distribution would be Gaussian. In fact, however, the data have an exponential distribution not only for r_1 but also for r_2, r_4, and r_8.

A second puzzle is the striking simplicity of the power law dependence of σ_1 on S_0. Such a result is reminiscent of critical phenomena and hints at the possibility of the economy self-organizing into a critical state.

4.1. *The exponential distribution of growth rates.* – The central limit theorem suggests that the distribution of T-year growth rates should be a Gaussian for T sufficiently large. However, the analysis of the data shows that eq. (5) is verified for $T \leqslant 8$, while for $T = 16$ the noise makes any interpretation difficult.

Thus, we can ask if there is a plausible modification of Gibrat's assumptions [14] that could lead to eq. (5). One possibility is to relax the assumption of uncorrelated growth rates and to assume that the successive growth rates are correlated in such a way that the size of a company is "attracted" to an optimal size S^*. This value may be interpreted as the minimum point of a "U-shaped" average cost curve in conventional economic theory and should evolve only slowly in time (on the scale of years) [47]. Let us then consider a set of companies all having initial sales S_0. As time passes, the sales of each of the firms will vary from day to day (or over another time interval much less than 1 year), but they tend to stay in the neighborhood of S^*. In the simplest case, the growth process has a constant "back-drift," *i.e.*

$$\frac{S_{t+\Delta t}}{S_t} = \begin{cases} k \exp[\varepsilon_t], & S_t < S^*, \\ \frac{1}{k}\exp[\varepsilon_t], & S_t > S^*, \end{cases} \tag{13}$$

where k is a constant larger than one and ε_t an uncorrelated Gaussian random number with zero mean and variance $\sigma_\varepsilon^2 \ll 1$. These dynamics are similar to what is known in economics as regression towards the mean [42, 43], although this formulation is not standard in economics.

Written in terms of the logarithmic growth rate $r_t \equiv \ln(S_t/S_0)$, eq. (13) reads

$$r_{t+\Delta t} - r_t = -\ln k \,\mathrm{sgn}(r_t - r^*) + \varepsilon_t, \tag{14}$$

where $r^* \equiv \ln(S^*/S_0)$ and $\mathrm{sign}(x) = -1$ for $x < 0$ and $\mathrm{sign}(x) = 1$ for $x > 0$.

For large times $t \gg \Delta t$ we can replace eq. (13) by its continuum limit and obtain

$$\Delta t \frac{\mathrm{d}r(t)}{\mathrm{d}t} = -\ln k \frac{\mathrm{d}}{\mathrm{d}r}|r(t) - r^*| + \sqrt{\Delta t}\varepsilon(t), \tag{15}$$

where now $\varepsilon(t)$ is a Gaussian random field with $\langle\varepsilon(t)\rangle = 0$ and $\langle\varepsilon(t)\varepsilon(t')\rangle = \sigma_\varepsilon^2 \delta(t-t')$ [48]. Here, $\langle\ldots\rangle$ means an average over realizations of the disorder. Equation (14) describes a strongly overdamped Brownian motion of a classical particle with mass 1 in a potential

$$V(r) = \ln k |r - r^*|, \tag{16}$$

where the friction constant is Δt and the thermal energy $\sigma_\varepsilon^2/2$ [49]. For large times t, the "particle coordinate" r is distributed according to the equilibrium Boltzmann distribution,

$$p(r_1 | s_0) = \frac{\ln k}{\sigma_\varepsilon^2} \exp\left[-\frac{2\ln k |r_1 - r^*|}{\sigma_\varepsilon^2}\right]. \tag{17}$$

Hence, we recover eq. (5) with $\bar{r}(s_0) = r^*$ and

$$\sigma_1(s_0) = \frac{\sigma_\varepsilon^2}{\sqrt{2}\ln k}. \tag{18}$$

The results expressed by eqs. (17)-(18) can account for the increase of σ_1 with the size of the company if we assume that σ_ε is a function of s_0. A model for such a dependence will be discussed in subsect. 4·3.

4·2. *Time dependence of the growth-rate distribution.* – Equation (17) describes the equilibrium distribution of the growth rates for sufficiently long times t. Our data suggests that σ_t grows with time, even for $t = 16$. One possible explanation is that we are still in the transient regime of the process in eq. (13). In order to find the distribution in the transient regime, we must write down the Fokker-Planck equation [49] associated with eq. (15):

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\sigma_\varepsilon^2}{\Delta t}\frac{\partial^2 f}{\partial r^2} + \frac{\ln k}{\Delta t}\frac{\partial f}{\partial r}\,\mathrm{sign}\,(r - r^*)\,. \tag{19}$$

Using dimensionless variables

$$x = \frac{r - r^*}{r_0}\,, \qquad u = \frac{t}{t_0}\,, \tag{20}$$

where

$$r_0 = \frac{\sigma_\varepsilon^2}{\ln k} \quad \text{and} \quad t_0 = \frac{\Delta t \sigma_\varepsilon^2}{(\ln k)^2}\,, \tag{21}$$

and imposing a mass-conservation condition

$$\int_0^\infty p(x)\,\mathrm{d}x = \int_{-\infty}^0 p(x)\,\mathrm{d}x = \frac{1}{2}\,,$$

we get the solution

$$p(x) = \frac{1}{\sqrt{2\pi u}}\,e^{-(|x| + u)^2/2u} + \frac{1}{2}\,\mathrm{erfc}\left(\frac{|x| - u}{\sqrt{2u}}\right)e^{-2|x|}\,, \tag{22}$$

which always satisfies the boundary condition

$$\left.\frac{\partial \ln p}{\partial x}\right|_{x = +0} = -2\,.$$

For large $u \gg x$, $u \gg 1$ ($t \gg r\Delta t/\ln k$, $t \gg \Delta t\sigma_\varepsilon^2/(\ln k)^2$), in agreement with eq. (16), the distribution can be well approximated by an exponential form

$$p(r|s_0) = \frac{1}{r_0}\,e^{-|r - r^*|/r_0}\,. \tag{23}$$

For small $u \ll x$ the slopes of the graphs of the $\ln p(x, u)$ can be well approximated by a linear equation $\partial \ln p/\partial x \approx -1 - x/u$, and thus the distributions $p(r_t, s)$ for large r are parabolas widening with the increase of t. The width of the distribution $p(x, u)$ is given by

$$\sigma_u^2 = \int_{-\infty}^{+\infty} p(x, u)\,x^2\,\mathrm{d}x = \frac{1}{2}\,\mathrm{erf}\sqrt{\frac{u}{2}} + \left(u + \frac{1}{2}u^2\right)\mathrm{erfc}\sqrt{\frac{u}{2}} - \frac{1}{\sqrt{2\pi u}}(u^2 + u)\,e^{-u/2}\,. \tag{24}$$

For small $u \ll 1$, σ_u^2 increases linearly with time, but for large u converges to its limiting value 1/2 in agreement with eq. (18). In terms of the original variable t, it happens when $t \gg t_0 = (\Delta t\sigma_\varepsilon^2)/(\ln k)^2$. The comparison of our experimental data with eqs. (22) and (24) suggest that these two equations correctly predict the qualitative behavior of $p(r_t, s_0)$ and σ_t, but fail to reproduce important quantitative details of the experimental data.

First, the distribution (22) for large x has a rate of decay faster than exponential while the real data has a rate of decay slower than exponential. Second, the distribution (22) always has a slope of -2 near the peak, while the slopes of the real graphs apparently decrease with time. Finally, the behavior of σ_t (24) has a sharp crossover at time t_0 from linear growth to constant, while the real data can be approximated as weak power law for long time spans. This means that for real data the transient time t_0 is very large.

These discrepancies can possibly be eliminated if one assumed that the noise ε in eq. (13) has long-range correlations $\langle \varepsilon_t \varepsilon_{t'} \rangle \sim |t - t'|^{-\gamma}$. Since the analytical solution of the problem is rather complicated, we attempted to solve the problem numerically, assuming for simplicity the Lévy walk [50] type of correlations. We simulate the multiplicative process described by eq. (13), assuming that companies undergo long periods of growth with positive $\varepsilon_t = +\varepsilon$, and long periods of recession with negative $\varepsilon_t = -\varepsilon$. The durations of these periods l we assume to be distributed according to a power law function

$$p(l) \sim l^{-\mu}, \qquad \mu = \gamma + 2 . \tag{25}$$

These long winning and losing streaks may represent either the general state of the economy of some catastrophic changes in firm size, *e.g.*, firm merging or splitting, events that do not happen instantaneously, but may, for large corporations, require a long transitional period of several years. In a logarithmic space, the processes of eq. (13) correspond to Lévy walks with unequal time steps: large steps directed toward the origin and small steps directed away from the origin. One can call this unusual type of motion a Lévy walk in a potential field.

It is well known that classical Lévy walks exhibit superdiffusive behavior when $\mu < 3$ [50]. Our numerical analysis suggests that in this case a Lévy walk in an attractive potential is not confined to the origin but σ_t diverges as power law

$$\sigma_t^2 = t^{4-\mu} . \tag{26}$$

This case clearly does not correspond to our experimental data, since σ_t^2 grows more slowly than t. On the other hand, when $\mu > 3$, Lévy walks are confined by the potential but have very large transient times t_0 which diverge as $\mu \to 3 + \varepsilon$. In this case, in the transient regime the distribution of growth rates have a tent-shaped form near the origin, but with power-law wings. Moreover, in this transient regime the slope of the tent shape decreases with time, and σ_t^2 grows approximately as small power of t, thus exactly reproducing all three unusual features of our experimental data. Hence Lévy correlated noise may provide a satisfactory explanation of our results. However, additional work is needed to examine other possibilities.

Another possible explanation for the time dependence of σ_T is that the optimal size of a company does not remain constant but, in fact, performs some sort of random walk with a very small diffusion coefficient $\mathscr{D}$. Such a model can be easily solved and it leads to the prediction that

$$p(s_T\,|\,s_0) = \frac{1}{2a}\,\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{-(s^*-s_0)^2/2t}\, e^{-(s_T-s^*)/a}\,\mathrm{d}s^*\,, \tag{27}$$

where $a = \sigma_1(s_0)/\sqrt{2}$ and $2\,t = \mathscr{D}T$. The analytical form of the distribution of growth rates is then given by

$$p(r_T\,|\,s_0) = \frac{1}{2a}\,e^{T/2a^2}\left[e^{-r_T/a}\,\mathrm{erfc}\left(\frac{t/a - r_T}{\sqrt{2t}}\right) + e^{r_T/a}\,\mathrm{erfc}\left(\frac{t/a + r_T}{\sqrt{2t}}\right)\right], \tag{28}$$

where $\mathrm{erfc}\,x = 2/\sqrt{\pi}\int_x^{+\infty}\exp[-y^2]\,\mathrm{d}y$. The total width of the distribution at time T is

$$\sigma_T^2 = 2a^2 + t = \sigma_1^2 + \mathscr{D}T\,. \tag{29}$$

Unfortunately, this result does not agree with the empirical data. Although the width of the distribution indeed increases with T, this increase is achieved by a rounding of the top of the distribution while the slope, on a linear-log plot, of the wings of the distribution remain constant. This prediction clearly disagrees with the observed change in the slope of the wings of the distribution for $1 \leqslant T \leqslant 8$. However the real phenomenon can be a joint effect of both correlated noise and changing of the optimal size.

4·3. *The scaling exponent* β. – While the model in the previous section explains eq. (5), it does not predict our finding about the the power law dependence of the standard deviation of growth rates on firm size. In this section, we show how a model of management hierarchies can predict eq. (7). In economics, it is generally presumed that the growth of firms is determined by changes in demand and production costs. Since these features are specific to individual markets, it is surprising that a law as simple as equation eq. (7) governs the growth rate of firms operating in much different markets. While demand and technology vary across markets, virtually all firms have a hierarchical decision structure. One possible explanation for why there is a simple law that governs the growth rate of all manufacturing firms is that the growth process is dominated by properties of management hierarchies [31]. This focus on the technology of management rather then technology of production as a basis for understanding firm growth is reminiscent of Lucas' model of the size distribution of firms [23].

At the outset let us acknowledge a tension between our empirical results and the theoretical model in this section. In the preceding sections, we analyze the scaling properties of the distribution of the logarithmic growth rate r_1 and its standard deviation σ_1. In this section we view companies as consisting of many business units. Since the sales of a company are the sum of the sales of individual units rather than their product, it is more convenient to analyze the standard deviation of the annual firm size change rather then the logarithmic growth rate. Let $\Sigma_1(S_0)$ be the standard deviation of end-of-period size for initial size S_0. Since $\sigma_1 \sim S_0^{-\beta}$ and since $S_1 \equiv$

$S_0 \exp[r_1] \approx S_0 + S_0 r_1$, it follows that $\Sigma_1(S_0) \approx S_0 \sigma_1 \sim S_0^{1-\beta}$. As discussed in Section III-D, σ_1 must be small for this approximation to hold.

Let us start by assuming that every company, regardless of its size, is made up of similarly sized units. Thus, a company of size S_0 is on average made up of $N = S_0/\bar{\xi}$ units, where

$$\bar{\xi} = \frac{1}{N} \sum_{i=1}^{N} \xi_i , \tag{30}$$

and ξ_i is the size of unit i. We further assume that the annual size change δ_i of each unit follows a bounded distribution with zero mean and variance Δ, which is independent of S_0. It is important to notice that throughout this section and the following we consider Δ"$\bar{\xi}^2$, to insure that sizes of units remain positive. Since some divisions after several cycles of growth may shrink almost to zero, while others grow several times, we assume that companies dynamically reorganize themselves so that they begin each period with approximately equal-sized divisions and the inequality $\Delta \ll \bar{\xi}^2$ holds.

If the annual size changes of the different units are independent, then the model is trivial. Using the fact that $\langle \delta_i \rangle = 0$, we have

$$\langle S_1 \rangle = S_0 + \sum_{i=1}^{N} \langle \delta_i \rangle = S_0 . \tag{31}$$

The second moment of the distribution is given by

$$\langle S_1^2 \rangle = \left\langle \left(S_0 + \sum_{i=1}^{N} \delta_i \right)^2 \right\rangle = S_0^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \langle \delta_i \delta_j \rangle = S_0^2 + N\Delta , \tag{32}$$

where we used again the fact that the δ_i's are centered and independent.

Thus, the variance in the size of the company is

$$\Sigma_1^2(S_0) = N\Delta = S_0 \frac{\Delta}{\bar{\xi}} \sim S_0 . \tag{33}$$

Using the fact that $\Sigma(S_0) \sim S_0^{1-\beta}$ (see subsect. 3·4), it follows that $\beta = 1/2$ [36].

The much smaller value of β that we find indicates the presence of strong positive correlations among a company's units. We can understand this result by considering the tree-like hierarchical organization of a typical company [31]. The head of the tree represents the head of the company, whose policy is passed to the level beneath, and so on, until finally the units in the lowest level take action. These units have again a mean size of $\bar{\xi} = S_0/N$ and annual size changes with zero mean and variance of Δ. Here we assume for simplicity that at every level other than the lowest each node is connected to exactly z units in the next lowest level. Then the number of units N is equal to z^n, where n is the number of levels (see fig. 7).

What are the consequences of this simple model? Let us first assume that the head of the company suggests a policy that could result in changing the size of each unit in

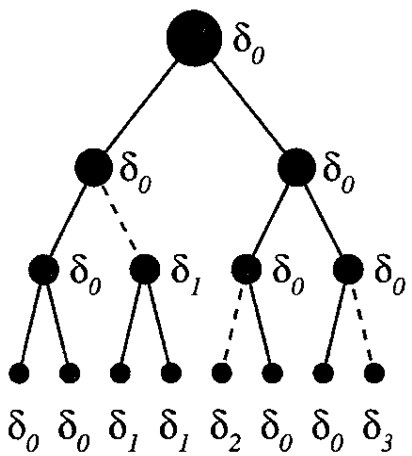

Fig. 7. – The hierarchical-tree model of a company. We represent a company as a branching tree. Here, the head of the company makes a decision about the change δ_0 in the size of the lowest level units. That decision is propagated through the tree. However, the decision is only followed with a probability Π. This is represent in the figure by a full link. With probability $(1-\Pi)$ a new growth rate is defined. This is represented in the figure by a slashed link. We see that at the lowest level there are clusters of values δ_i for the changes in size.

the lowest level by an amount δ_0. If this policy is propagated through the hierarchy without any modifications, then it is the same as assuming in eq. (4) that all the δ_i's are identical. This implies that

$$\langle S_1^2 \rangle = S_0^2 + N^2 \Delta \,, \tag{34}$$

from which follows

$$\Sigma_1^2(S_0) = N^2 \Delta = S_0^2 \frac{\Delta}{\xi^2} \,, \tag{35}$$

and we conclude that $\beta = 0$.

Of course, it is not realistic to expect that all decisions in an organization would be perfectly coordinated as if they were all dictated by a single "boss." Hierarchies might be specifically designed to take advantage of information at different levels; and mid-level managers might even be instructed to deviate from decisions made at a higher level if they have information that strongly suggests that an alternative decision would be superior. Another possible explanation for some independence in decision-making is organizational failure, due either to poor communication or disobedience.

To model the intermediate case between $\beta = 0$ and $\beta = 1/2$, let us assume that the head of a company makes a decision to change the size of the units of a company by an amount δ_0. We also assume that δ_0, for the set of all companies, has zero mean and variance Δ. Furthermore, we consider that each manager at the nodes of the hierarchical tree follows his supervisor's policy with a probability Π, while with probability $(1-\Pi)$ imposes a new *independent* policy. The latter case corresponds to the manager acting as the head of a smaller company made up of the units under his supervision. Hence the size of the company becomes a random variable with a standard deviation that can be computed either with numerical simulations or using recursion relations among the levels of the tree.

The proposed model is analogous to the expansion modification models used by Li to explain long-range correlations in the DNA sequences [51] and allows a simple analytical solution. In fact, the local production units with numbers l and $l+k$, where k is the large number, are connected to each other through $\log_z k$ levels of firm hierarchy. Thus the correlations among them are equal to $\Pi^{2\log_z k}$, since it is required that $\log_z k$ links going up and $\log_z k$ links going down to connect them. Thus correlations between production units decay as $k^{2\ln\Pi/\ln z}$. The variance Σ_1^2 of the total size of N production units is thus

$$\Sigma^2 \sim N^{2+2\ln\Pi/\ln z} \sim S_0^{2+2\ln\Pi/\ln z}, \tag{36}$$

which implies $\beta = -\ln\Pi/\ln z$. If $-\ln\Pi/\ln z \geqslant 1/2$, the units become uncorrelated on large scales and Σ^2 grows as S_0, which implies $\beta = 1/2$.

Finally, we can write, for $n \gg 1$, that the hierarchical model leads to

$$\beta = \begin{cases} -\ln\Pi/\ln z & \text{if } \Pi > z^{-1/2}, \\ 1/2 & \text{if } \Pi < z^{-1/2}. \end{cases} \tag{37}$$

Even for small n, we find that eq. (37) is a good approximation—*e.g.*, while for $z=2$ and $\Pi = 0.87$ we predict $\beta = 0.20$, when we take $n=3$ the deviation from the predicted value is only 0.03, *i.e.*, about 15%.

Equation (37) is confirmed in the two limiting cases: when $\Pi = 1$ (absolute control) $\beta = 0$, while for all $\Pi < 1/z^{1/2}$, decisions at the upper levels of management have no statistical effect on decisions made at lower levels, and $\beta = 1/2$. Moreover, for a given value of $\beta < 1/2$ the control level Π will be a decreasing function of z: $\Pi = z^{-\beta}$, cf. fig. 8. For example, if we choose the empirical value $\beta \approx 0.15$, then eq. (37) predicts the plausible result $0.9 \geqslant \Pi \geqslant 0.7$ for a range of z in the interval $2 \leqslant z \leqslant 10$.

Our data for σ_T suggests that for larger time intervals β decreases. Can this be explained within the framework of the hierarchical model? The answer is yes. The decrease in β with time suggests that the activity of the company becomes more coordinated on large time scales. It means that the probability Π increases with time. This is very plausible, since the information may propagate through the hierarchical structure of the company with finite speed. On small time scales, the activity of the local manager is less coordinated with the general policy of the company headquarters. For example, firing and hiring small numbers of employees may be completely the responsibility of local managers. A major decision, *e.g.*, the firing of a large number of employees, made at the top of the hierarchy is a relatively infrequent event (on a time

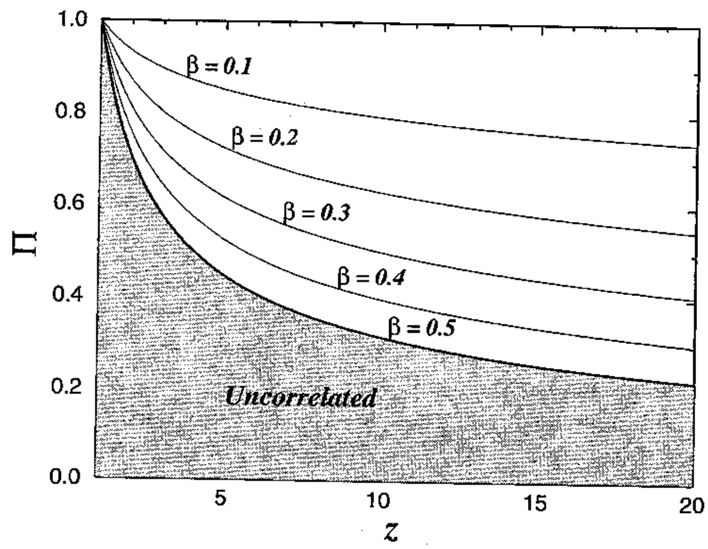

Fig. 8. – Phase diagram of the hierarchical-tree model. To each pair of values of (Π, z) corresponds a value of β. We plot the iso-curves corresponding to several values of β. In the shaded area, marked "Uncorrelated," the model predicts that $\beta = 1/2$, *i.e.* that the units of the company are uncorrelated. Our empirical data suggests that most companies have values of Π and z in between the curves for $\beta = 0.1$ and $\beta = 0.2$.

scale of several years), but when it does occur, it is enforced strictly throughout all levels of the hierarchy.

4·4. *Combining the two models.* – We started with two central empirical findings about firm growth rates. The model in sect. 2 predicts one of those findings (the shape of the distribution) and the model in sect. 3 predicts the other (the power law dependence of the standard deviation of output on firm size). This section addresses the relationship between the two models. First, we address concerns that the models might be contradictory and show that they are not. Then, we show how the models can be combined into a single model that predicts both of our empirical findings.

In the tree model, firm growth rates are potentially the result of many independent decisions. As a result, one might expect that the central limit theorem would imply a Gaussian distribution of firm output. In fact, however, the distribution of outputs is not necessarily Gaussian.

To address the distribution of firm output in the tree model, it is necessary to make an assumption about the distribution from which each independent growth decision is drawn. No such assumption is needed to analyze the standard deviation of firm growth rates, but is needed to analyze the shape of the distribution.

In fig. 9, we show the distribution of the inputs (*i.e.* of each independent decision) and the outputs for a tree with $z = 2$, $\Pi = 0.87$, and $n = 10$. We find that for Gaussian

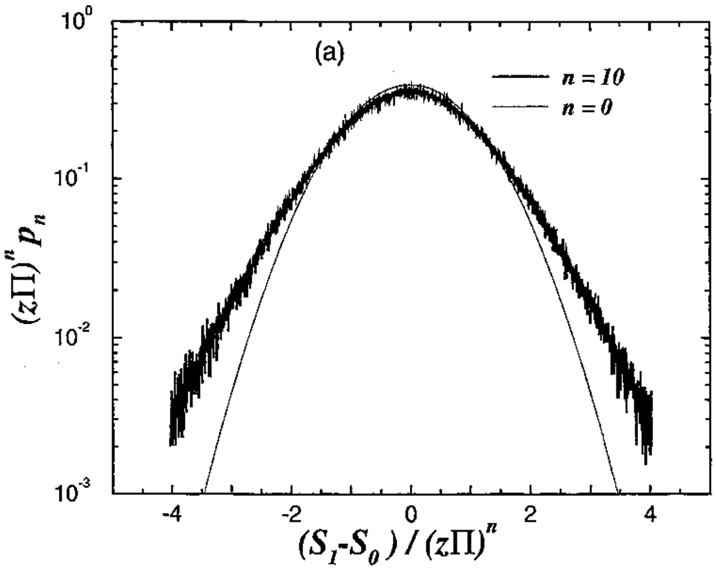

Fig. 9a. – Probability density for the output and input variables in the tree model. Here we have $z = 2$, $\Pi = 0.87$, and $n = 10$. Gaussian distribution of the input.

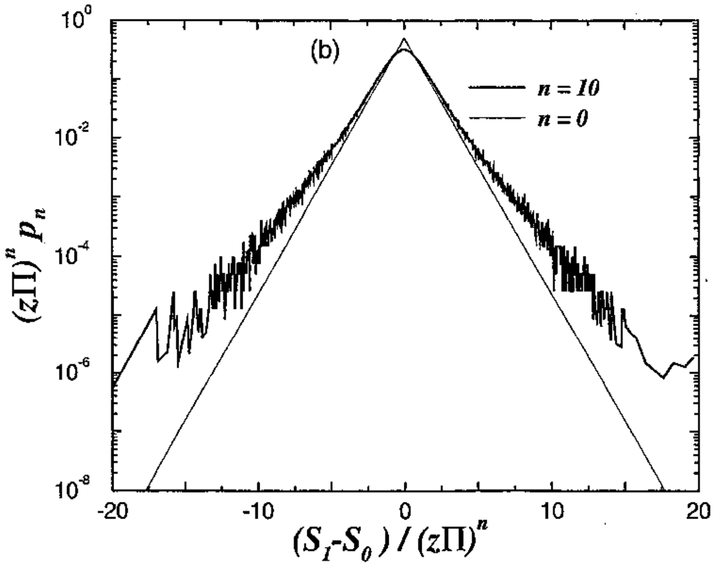

Fig. 9b. – Probability density for the output and input variables in the tree model. Exponential distribution of the input with the same parameters as in fig. 9a.

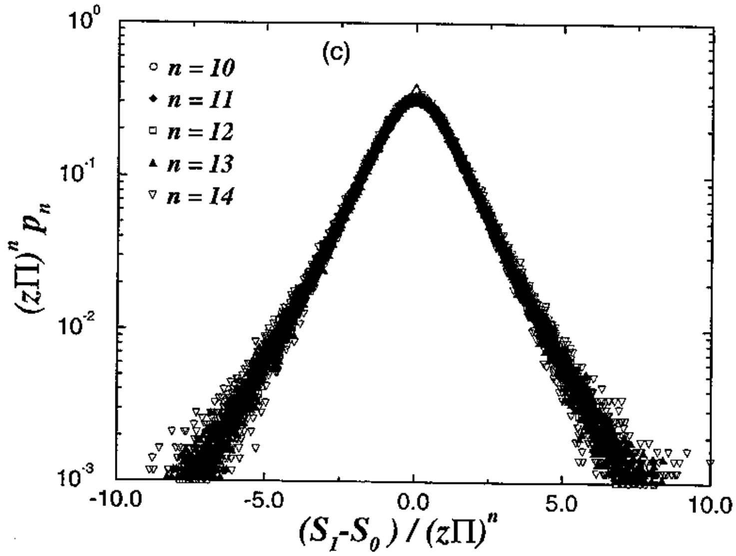

Fig. 9c. – Data collapse of the output distribution for trees with different number of levels. The other parameters remain unchanged and the input is exponentially distributed. It is visually apparent the similarity of the numerical results with the empirical data of fig. 6a.

distributed inputs, the output is not Gaussian in the tails. This finding is remarkable. First of all, with $z = 2$ and $n = 10$, the firm consists of 1024 units. With a probability to disobey of $1 - 0.87 = 0.13$, one would expect $0.13 \times 1024 \approx 133$ of the units to, on average, make independent decisions about their growth rates. Thus, even for non-Gaussian inputs, one can hypothesize that the output is close to Gaussian. Moreover, for Gaussian inputs, the sum of independent Gaussians is itself Gaussian. Thus, for every particular configuration of the disobeying links, the output distribution is Gaussian with variance $m\Delta$, which is a function of this random configuration. However, there are $2^{(z^{n+1}-z)/(z-1)}$ possible configurations of links each of which produce a Gaussian distribution with different integer m. Figure 10 shows the probability p_m^n to get a tree with given m computed for all trees with a given number of levels n, $\Pi = 0.87$, and $z = 2$. As visually apparent in fig. 10, this probability density is a non-trivial function, the discussion of the analytical properties of which is beyond the scope of the present article. The final distribution of the firm output S_1 will be thus given by the convolution of two densities: p_m^n and Gaussian with variance $m\Delta$

$$p_n(S_1) = \sum_m p_m^n \frac{1}{\sqrt{2\pi m\Delta}} e^{-(S_1-S_0)^2/2m\Delta}, \tag{38}$$

which is no longer Gaussian for the observed form of p_m^n.

In a general case, it can be shown by martingale theory [52] that for any input distribution $f(x)$ with zero mean and finite variance Δ, the output distribution

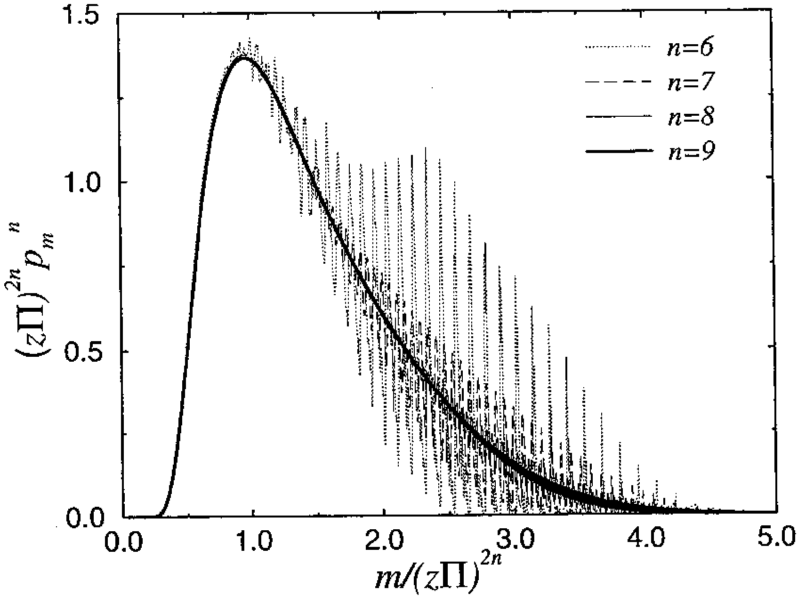

Fig. 10. – Numerical estimation from exact enumeration of the probability p_m^n to find a tree with η levels of hierarchy and a given variance $m\Delta$ of the distribution of its annual size change, provided that $\Pi = 0.87$ and $z = 2$. It is visually apparent that p_m^n converges to a smooth scaling function when $n \to \infty$.

converges for $n \to \infty$ to a distribution

$$\frac{1}{\sum\limits_1(n)}\, g_f\!\left(\frac{x}{\sum\limits_1(n)}\right), \tag{39}$$

where g_f is a function that does not depend on n but depends on f. Thus, we cannot expect to obtain a result that the output distribution must be exponential regardless of the input distribution. It would, however, be desirable to find some simple input distribution that yields the output distribution that we actually observe. Figure 9 also shows the output distribution when the input distribution is exponential in terms of $S_1 - S_0$. For small σ_1, it practically coincides with eq. (17). In this case, the output distribution is nearly exponential, and the slightly fatter wings that we observe are arguably consistent with our empirical results. Thus, in the limit of small σ_1, we can combine the models of the two sections by assuming that the dynamic process described in sect. 2 provides the input distribution for the tree model in sect. 3. This additional assumption in the tree model then predicts both of our empirical findings. For large σ_1, the direct combination of two models needs additional fine-tuning.

5. – Conclusions

In summary, we study publicly-traded US manufacturing companies from 1974 to 1993. We find that the distribution of the logarithms of the growth rate decays exponentially. Furthermore, we observe that the standard deviation of the distribution of growth rates scales as a power law with the size S of the company, and grows slowly with time T. We propose new models that give some insight into these results. We solve the models both numerically and analytically.

The models proposed are quite elementary, and show that simple mechanisms may provide some insight into our findings. Our central results, eqs. (3.5) and (3.7), constitute a test that any accurate theory of the firm must pass, and support the possibility that the scaling laws used to describe complex but inanimate systems comprised of many interacting particles (as occurs in many physical systems) may be usefully extended to describe complex but animate systems comprised of many interacting subsystems (as occurs in economics). Furthermore, the kind of scaling laws found in this study can be viewed as empirical evidence supporting some hypothesis regarding the self-organization of the economy [53].

This paper also proposes to offer a new approach to the study of economics. In this approach, numerical results are obtained from the empirical data and lead to numerical tests of the models. The degree of accuracy to which the empirical results can be repreduced with simple (and maybe simplistic) models shows the possibilities of this approach. To develop the models in the direction of greater realism, we are pursuing the analysis of the empirical data. Possible directions of research include studying data from different countries to see if the results (namely the value of the exponents) depend, *e.g.*, on different national legislation.

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