

Spatiotemporal fluctuations in growth phenomena: Dynamical phases and $1/f$ noise

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We find that the dynamics of growth phenomena exhibits the feature that growing sites are naturally partitioned into two “dynamical phases,” an *extinct* phase (the sites at which no further growth occurs, in a statistical sense) and a *surviving* phase (where growth occurs at some later time). As examples, we treat two common growth phenomena, invasion percolation and diffusion-limited aggregation. For both examples, we find “subclusters” which exhibit spatio-temporal fluctuations that are scale invariant. We find a $1/f^\varphi$ power spectrum for both models. Also, we identify an order parameter and find an associated exponent β .

Despite all the activity in recent years on the topic of disorderly growth phenomena, the existing *temporal* fluctuations of the actual growth process have not been addressed,¹ much less understood. Rather, the emphasis has been placed on the structure and its characterization by various fractal exponents. One possible reason for this state of affairs is that “mass=time”—i.e., a new particle is added at each tick of a clock. As a result, it is assumed that the temporal evolution has no particular information of physical interest. Indeed, experimentally observed physical phenomena such as $1/f$ noise have *not* been associated with growth models, and studies of such temporal fluctuations are restricted to models of a rather different character. In this Rapid Communication we propose an altogether new approach to understanding spatiotemporal fluctuations in growth phenomena. We illustrate our method by explicit calculations for two of the most-studied prototype models in growth phenomena, invasion percolation² and diffusion-limited aggregation (DLA).^{3,4}

Our key result is the identification of a natural process by which the dynamics of the growth phenomenon itself segregates *extinct* (i.e., inactive, in a statistical sense) subclusters. We find that these subclusters exhibit spatiotemporal fluctuations that are scale invariant; in particular, we find that invasion percolation exhibits $1/f^\varphi$ noise with φ near unity, while for DLA $\varphi \approx 2$. Moreover, we identify the associated order parameter j as the *surviving* (active) fraction of the cluster, and introduce a corresponding order-parameter exponent β . We calculate β and compare our results with the prediction of a “mean-field”-type approach.⁵

Consider first the invasion percolation model, without trapping. First one assigns to every site of a lattice a random number between zero and one. At time 1 the origin is occupied, but at each successive time interval growth occurs at the perimeter site with the lowest random number. Figure 1(a) shows a cluster of “mass” $M=4000$ grown on a square lattice. As the cluster grows, the normalized distribution of random numbers $\mathcal{N}(p)$ on the perimeter sites will change. Since always the *smallest* random number is becoming a part of the cluster, the distribution will favor *large* random numbers. In the infinite

mass limit the dynamics gives rise to a “critical probability” p_c at which $\mathcal{N}(p)$ jumps from *zero* to a nonzero value of $1/(1-p_c)$.

We propose that perimeter sites may be partitioned into two “dynamical phases,” an *extinct phase* where are the sites at which no further growth occurs, and a *surviving phase* where growth occurs at some later time. The partitioning of clusters is based on the question where, for example, the flow in fluid displacement actually takes place. In the infinite mass limit these two phases are separated by p_c : the extinct phase has $p > p_c$, the surviving phase has $p < p_c$. However, at *finite* sizes we can partition the perimeter sites into those sites with $p > p^*$ (extinct sites) and those with $p \leq p^*$ (surviving sites). Here p^* is defined by⁶

$$p^* \equiv 2\langle p \rangle - 1, \quad (1a)$$

where

$$\langle p \rangle \equiv \int_0^1 p \mathcal{N}(p) dp. \quad (1b)$$

Note that in the infinite mass limit, $\langle p \rangle = (1+p_c)/2$, and $p^* = p_c$. However, at finite sizes, p^* does not equal p_c .

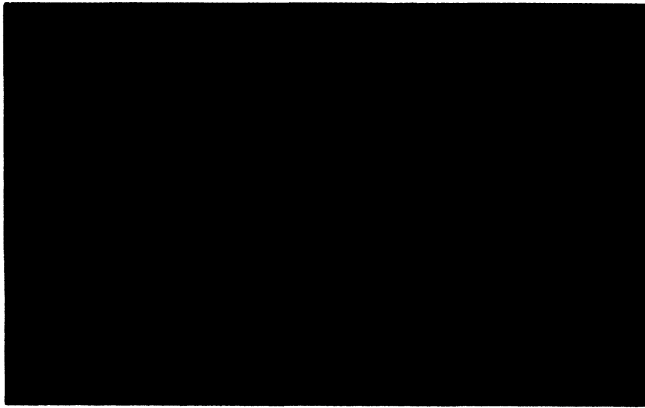
Each surviving site can be traced *back* to the original seed, choosing at each step the neighbor site which was grown most recently. We call the paths thereby obtained *surviving paths*. If all the surviving paths are removed from the cluster, the cluster splits into “subclusters” [Fig. 1(a)]. Each subcluster is thus an *extinct branch* of mass s created from a “perturbation” on a surviving path. The surviving paths are physically distinct from the extinct branches in the sense that they “carry the flow.”

To introduce the concept of time, we associate with each cluster site a number giving the order in which the site becomes part of the cluster. Then the *lifetime* t of an extinct branch is the difference between the first site and the last site on that branch.⁷

Figures 1(b) and 1(c) show the distribution $D(s)$ of sizes and $D(t)$ of lifetimes. We find that both distributions follow power laws

$$D(s) \sim s^{1-\tau}, \quad (2a)$$

$$D(t) \sim t^{\varphi-2}, \quad (2b)$$



(a)

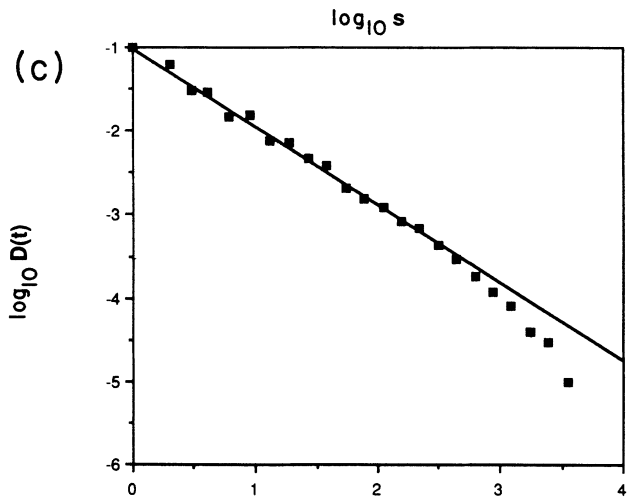
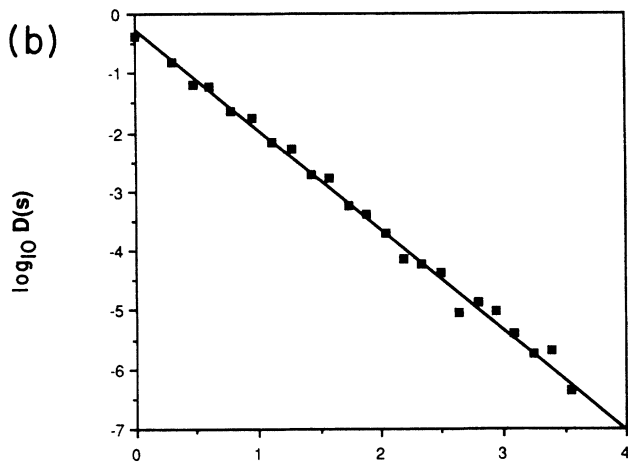


FIG. 1. (a) Invasion percolation cluster of mass $M=4000$. The surviving paths are shown as heavy lines. The remainder of the cluster is a disconnected set of extinct branches. The surviving perimeter sites are shown in green, the surviving paths in blue, and the extinct branches in red. (b) The size distribution $D(s)$ of extinct branches. (c) The distribution $D(t)$ of lifetimes t . The slopes of the straight lines are -1.69 and -0.93 , respectively. The distributions are based on 40 clusters of mass $M=4000$ (totaling ~ 6000 extinct branches). The straight line fits to the data include points with $3 \leq s \leq 362$ ($3 \leq t \leq 362$).

with $\tau \approx 2.69$ and $\varphi \approx 1.07$. The finite size does not appear to influence $D(s)$ below $s = s_c \approx M$. In contrast, the finite-size effect is apparent in $D(t)$ for large t . Since $s_c \leq t_c \leq M$, it follows that $t_c \sim M$.⁸ By calculating $D(t)$ for other values of mass, we have confirmed this cross-over scaling.

The perturbations may be interpreted as a “noise” on the surviving paths. Since the distribution of lifetimes



(a)

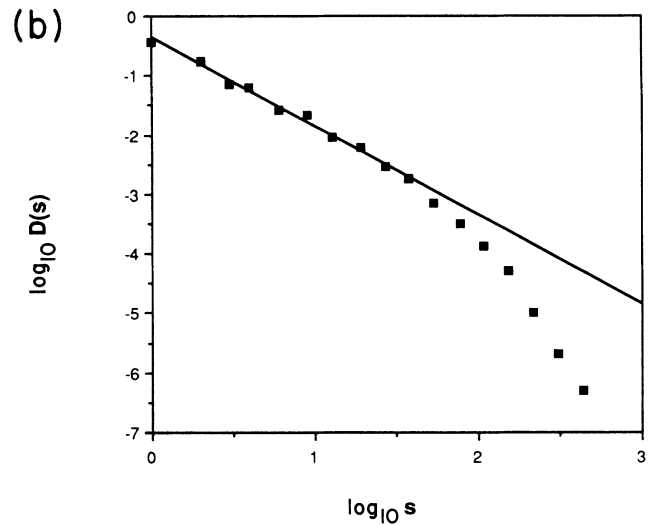


FIG. 2. (a) DLA cluster of mass $M=4000$. The surviving paths are shown as heavy lines. The rest of the cluster is a disconnected set of extinct branches. The surviving perimeter sites are shown in green, the surviving paths in blue, and the extinct branches in red. (b) The size distribution $D(s)$ of extinct branches, based on 40 clusters of mass $M=4000$ (totaling ~ 14000 extinct branches). The slope of the straight line is -1.50 and is fitted to the data for $3 \leq s \leq 45$. (c) $D(s)s^{1.5}$ vs sp^* based on 20 clusters of mass $M=400$ (\times), $M=1000$ (\square), and $M=4000$ (\blacktriangle). The data obtained for 40 clusters with mass $M=4000$ are also given (\triangle) for comparison. (d) $MD(t)$ vs t/M , symbols as in (c). The large scatter at small t is an effect of the broad distribution that only leaves a few number of lifetimes in the first “bins.”

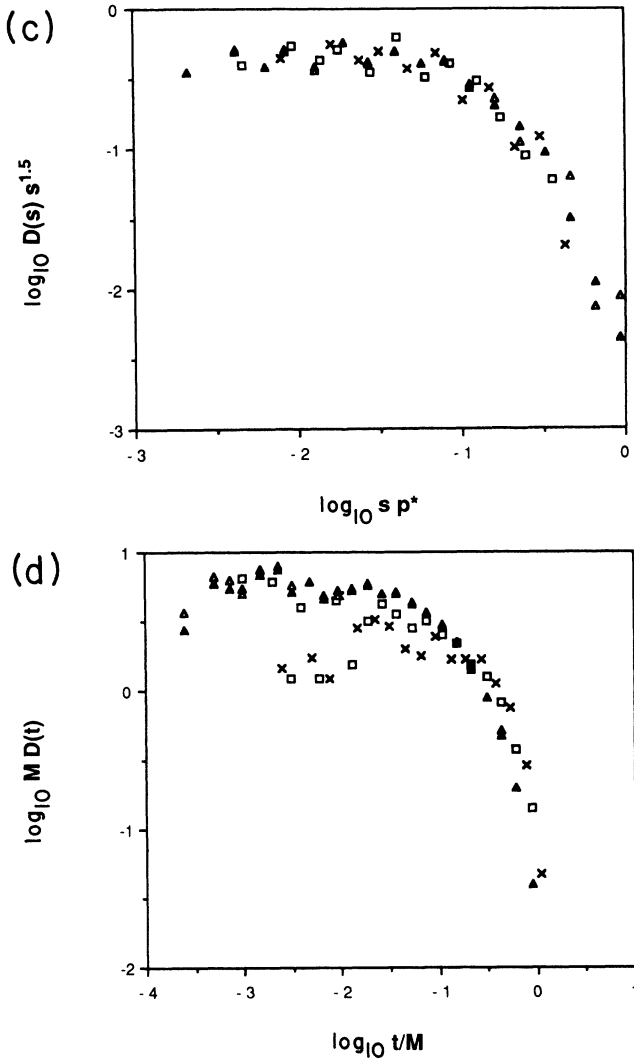


FIG. 2. (Continued).

(2b) translates directly into the power frequency spectrum,

$$P(f) \equiv \int_0^{1/f} tD(t)dt \sim f^{-\varphi}, \tag{3}$$

the result $\varphi=1$ would imply that the noise is $1/f$ noise. Next consider DLA.³ One releases a random walker from a random point on a circle surrounding the cluster, and growth takes place when the random-walker steps onto a perimeter site.⁹ Figure 2(a) shows a typical cluster of mass $M=4000$ grown on a square lattice. Using the electrostatic formulation,⁴ the growth probabilities along the perimeter are determined by solving the Laplace equation $\nabla^2\phi=0$ where the potential ϕ satisfies $\phi=0$ on the cluster and $\phi=1$ on the surrounding circle. At the perimeter, the growth probabilities p_i are proportional to the field, $p_i \sim |\nabla\phi_i|$.

To identify the *surviving sites* we note that the analog for DLA of p^* can be found from the spectrum $f(\alpha) \equiv \ln N(\alpha)/\ln L$, where $2L$ is the diameter of the cluster and $N(\alpha)d\alpha$ is the number of exponents

$\alpha \equiv -\ln p/\ln L$ in the range $[\alpha, \alpha+d\alpha)$. Thus, $N(\alpha) \sim L^{f(\alpha)}$ and $p \sim L^{-\alpha}$, and the probability $P(\alpha)d\alpha$ that the random walker steps on a perimeter site with the exponent α is given by

$$P(\alpha) = N(\alpha)p \sim N(\alpha)L^{-\alpha} \sim L^{f(\alpha)-\alpha}. \tag{4}$$

Since the probability is unity that the next growth occurs at *some* site,

$$\int P(\alpha)d\alpha = 1. \tag{5a}$$

Integrating (4), we find

$$1 = \int P(\alpha)d\alpha \sim \int L^{f(\alpha)-\alpha}d\alpha \sim L^{f(\alpha^*)-\alpha^*}, \tag{5b}$$

where α^* is determined by steepest descent, $f'(\alpha^*)=1$. Hence from (5a) and (5b), $f(\alpha^*)=\alpha^*$, while $f(\alpha)$ must be less than α for all other values of α . From (4), $P(\alpha)$ is a delta function, $P(\alpha)=\delta(\alpha-\alpha^*)$ in the large- L limit. Thus for a finite system,

$$\begin{aligned} \alpha^* &= \int \alpha P(\alpha)d\alpha = \int \left[-\frac{\ln p}{\ln L} \right] N(\alpha)p d\alpha \\ &= \sum_i \left[-\frac{\ln p_i}{\ln L} \right] p_i. \end{aligned} \tag{6a}$$

Defining p^* through $\alpha^* \equiv -\ln p^*/\ln L$, we obtain

$$\ln p^* = \sum_i p_i \ln p_i, \tag{6b}$$

which expresses p^* as the *dominating* probability.

We now identify the surviving sites for DLA as those perimeter sites with growth probabilities $p \geq p^*$. Again, these sites are traced back to the seed to find the surviving paths [Fig. 2(a)] and, removing these, we obtain a set of extinct branches. Figure 2(b) shows the size distribution $D(s)$, from which we find $\tau \approx 2.50$. The largest s is nearly 10 times smaller than that of invasion percolation. This gives a smaller scaling regime and a more pronounced crossover. To demonstrate that the falloff is due to the finite-size effect, we have also evaluated the distributions $D(s)$ for $M=400$ and $M=1000$. Figure 2(c) shows $D(s)s^{\tau-1}$ versus sp^* . The data collapse implies that $D(s)$ is well described by the scaling form

$$D(s) = s^{1-\tau} g(s/s_c), \tag{7a}$$

where $s_c \sim p^{*-1/\sigma}$, with $\sigma \approx 1$ and $g(x)$ a constant for $x \ll 1$.

We also calculate $D(t)$ for DLA; from (3) we find $\varphi \approx 2$. A data collapse for $D(t)$ of different cluster masses yields the scaling form [Fig. 2(d)]

$$D(t) = h(t/t_c)/t_c, \tag{7b}$$

with $t_c \sim M$ and $h(x)$ constant for $x \ll 1$. The long surviving times for DLA correspond to a *large* number of surviving paths [Fig. 2(a)], compared to *essentially one* surviving path in invasion percolation [Fig. 1(a)].

Next we discuss the *order parameter*. For both invasion percolation and DLA clusters we may define the “deviation from criticality”

$$\epsilon \equiv |p^* - p_c|, \quad (8a)$$

where $p_c = 0$ for DLA since the growth probabilities converge to zero in the infinite mass limit. As the cluster grows, $\epsilon \rightarrow 0$, and the order parameter is

$$j(\epsilon) \equiv N_s(\epsilon)/M \sim \epsilon^\beta, \quad (8b)$$

where N_s is the number of surviving perimeter sites.^{10,11}

To evaluate β for invasion percolation, note¹² that the excess areas under $\mathcal{N}(p)$ on both sides of p_c scale as $M^{-\sigma}$. Hence $j \sim M^{-\sigma}$ and $|p^* - p_c| \sim M^{-\sigma}$. Hence

$$\beta = 1. \quad (9a)$$

For DLA, β is related to the fractal dimension D . Since $p^* \sim L^{-\alpha^*} \sim M^{-\alpha^*/D}$, and the number of surviving sites $N_s \sim L^{f(\alpha^*)} \sim p^{*-\mathcal{R}}$, where $\mathcal{R} \equiv f(\alpha^*)/\alpha^*$, (8b) yields $j \sim p^{*-R}/p^{*-D/\alpha^*} \sim p^{*(D/\alpha^*)-R}$. At large values of L , $\mathcal{R} \rightarrow 1$. Since $P(\alpha^*) = \delta(\alpha - \alpha^*)$, α^* is also the information dimension. For DLA it is generally accepted that $\alpha^* = 1$.¹³ Hence¹⁴

$$\beta = D - 1. \quad (9b)$$

Finally, we extract the scaling behavior for the total number of extinct branches $\mathcal{N}_e(M)$ on a cluster of mass M , based on the observation that essentially all branches become extinct ($j \rightarrow 0$). This implies

$$\mathcal{N}_e(M) \langle s \rangle \sim M, \quad (10)$$

where $\langle s \rangle \equiv \int s D(s) ds$, and the left-hand side is the total mass of the extinct branches. Since $s_c \sim M$ for invasion percolation, and $\langle s \rangle \sim s_c^{3-\tau}$,

$$\mathcal{N}_e(M) \sim M^y, \quad (11)$$

with $y = \tau - 2 \simeq 0.69$. For DLA, $\langle s \rangle \sim p^{*-y}$, where $y = (3 - \tau)/\sigma \simeq 0.50$. Since $p^* \sim M^{-1/D}$, $y = 1 - \gamma/D \simeq 0.71$. The scaling relation $\beta = (\tau - 2)/\sigma$ does not apply

here. We can, however, think of the order parameter as originating from clusters larger than a certain size $s_0 \sim |p^* - p_c|^{-1/\sigma_0}$, where $\sigma_0 = (\tau - 2)/\beta$. For invasion percolation, $\sigma_0 = y$. Surprisingly, we find this relation to hold for DLA as well.

We can compare our results to the prediction $\tau = \frac{5}{2}$ and $\phi = 1$ in the “mean-field” limit, in which growth in different regions is statistically independent.⁵ Consistent with a “burst” dynamics,² we find that invasion percolation behaves *temporally* mean-field-like. Conversely, the ramification in DLA manifests in a *spatial* mean-field behavior.

In conclusion, we have proposed a framework for quantifying spatiotemporal fluctuations in growth phenomena. We find that perturbations grow with all time and length scales. The fluctuations are described by a new set of critical exponents, which we have determined for invasion percolation and DLA. The burst dynamics of invasion percolation and the ramification of DLA correspond to exponents that are close to the values for a general noninteracting branching process.

Experimentally, surviving sites are found by observing the interface over a small time period.¹⁵ If the structure is ramified, the surviving paths can be determined and the extinct branches obtained. For experiments in the DLA universality class, p^* can also be estimated since the small growth probabilities do *not* contribute to p^* , and the large probabilities are well approximated by the local growth. Aside from geometrical methods, the critical exponents τ and φ can be experimentally measured by directly observing the spatial and temporal fluctuations.¹⁶ The noise spectrum can also be obtained experimentally from the temporal behavior of velocity or pressure.

We thank the National Science Foundation and the Office of Naval Research for financial support.

¹Temporal fluctuations in the context of noise at a diffusion front has recently been studied in J. F. Gouyet, B. Sapoval, Y. Boughaleb, and M. Rosso, *Physica A* **157**, 620 (1989).

²See, e.g., L. Furuberg, J. Feder, A. Aharony, and T. Jøssang, *Phys. Rev. Lett.* **61**, 2117 (1988), and references therein.

³T. A. Witten and L. M. Sander, *Phys. Rev. Lett.* **47**, 1400 (1981).

⁴L. Niemeyer, L. Pietronero, and H. J. Wiesmann, *Phys. Rev. Lett.* **52**, 1033 (1984).

⁵P. Alstrøm, *Phys. Rev. A* **38**, 4095 (1988).

⁶Of course, for any finite size, no matter how large, some of the extinct sites will eventually grow at a later time, and some of the surviving sites will never grow. We emphasize, however, that the fraction \mathcal{A} of such “maverick” sites converges to zero, $\mathcal{A} \sim M^{-\sigma}$, where $\sigma \simeq 0.4$ (Ref. 12). Hence, these sites have no statistical relevance.

⁷Again, although growth occurs of some of the extinct branches, the statistical properties will not be influenced, since $\mathcal{A} \rightarrow 0$ (cf. Refs. 6 and 12). A similar definition appears in S. Havlin, R. Nossal, and B. Trus, *J. Phys. A* **17**, L957 (1984).

⁸This is consistent with a dynamical exponent $z = D$ ($t_c \sim L^z$, cf. Ref. 2).

⁹This boundary condition is typical for dielectric breakdown models (Ref. 4).

¹⁰That the growth process only *barely* survives is one of the characteristic features of self-organized critical phenomena (Ref. 11), where the “flow” j plays the role of an order parameter.

¹¹P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987); C. Tang and P. Bak, *Phys. Rev. Lett.* **60**, 2347 (1988); P. Alstrøm, *Phys. Rev. A* (to be published).

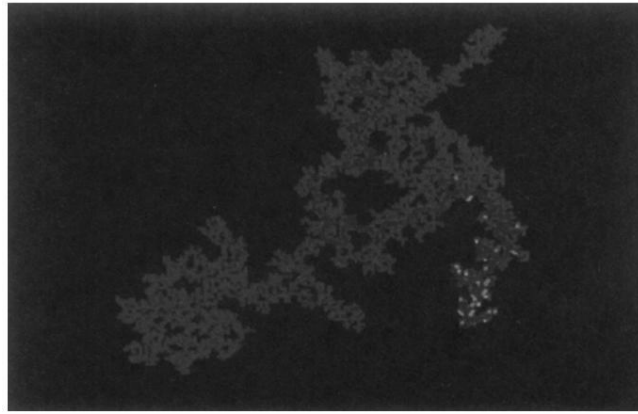
¹²D. Wilkinson and J. F. Willemsen, *J. Phys. A* **16**, 3365 (1983).

¹³See, e.g., T. C. Halsey, P. Meakin, and I. Procaccia, *Phys. Rev. Lett.* **56**, 854 (1986).

¹⁴For finite clusters \mathcal{R} is always smaller than 1, and \mathcal{R} expresses how well p^* dominates the growth. We plotted j vs p^* and N_s vs p^* and found $\beta \simeq 0.8$, and $\mathcal{R} \simeq 0.9$. These values are consistent with $D \simeq 1.7$. Also, D can be determined indirectly by plotting M versus p^* . By this method, we find $D = 1.73$. Note that a numerical determination of the length scale L of the clusters is *not* necessary to obtain β , \mathcal{R} , or D .

¹⁵Y. Couder and T. Jøssang (private communication).

¹⁶When a new branch is created or a branch “dies,” it gives rise to a small change in volume, as well as in velocity and pressure.



(a)

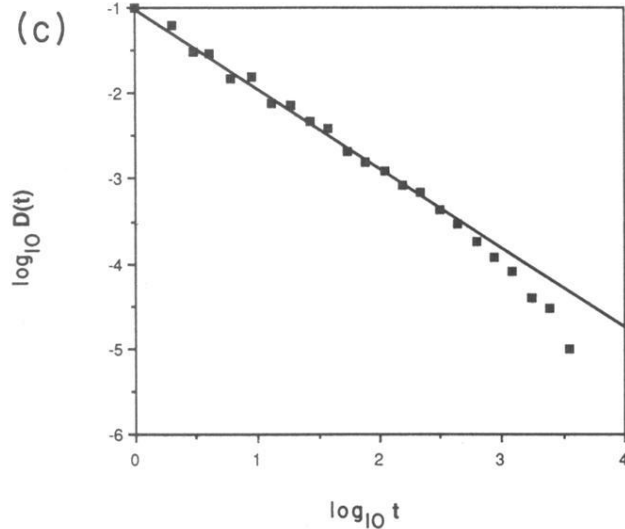
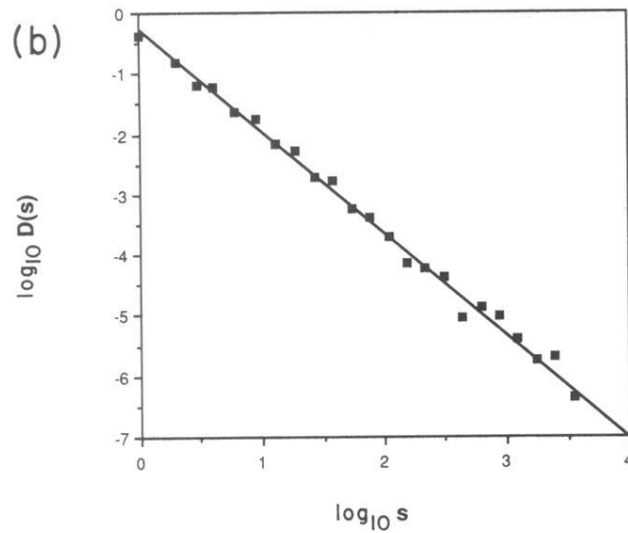
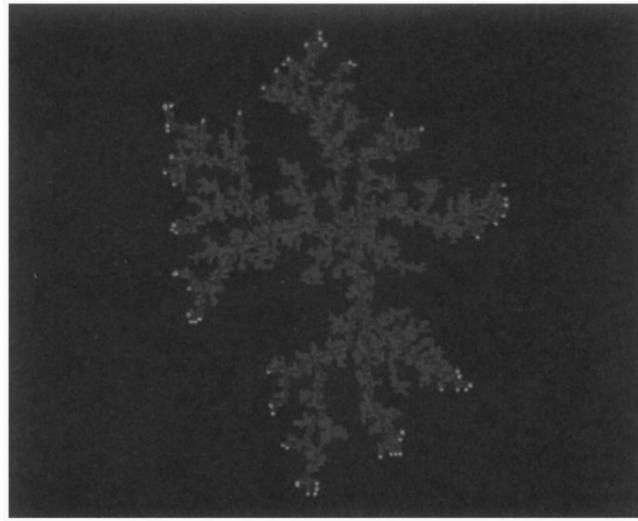


FIG. 1. (a) Invasion percolation cluster of mass $M=4000$. The surviving paths are shown as heavy lines. The remainder of the cluster is a disconnected set of extinct branches. The surviving perimeter sites are shown in green, the surviving paths in blue, and the extinct branches in red. (b) The size distribution $D(s)$ of extinct branches. (c) The distribution $D(t)$ of lifetimes t . The slopes of the straight lines are -1.69 and -0.93 , respectively. The distributions are based on 40 clusters of mass $M=4000$ (totaling ~ 6000 extinct branches). The straight line fits to the data include points with $3 \leq s \leq 362$ ($3 \leq t \leq 362$).



(a)

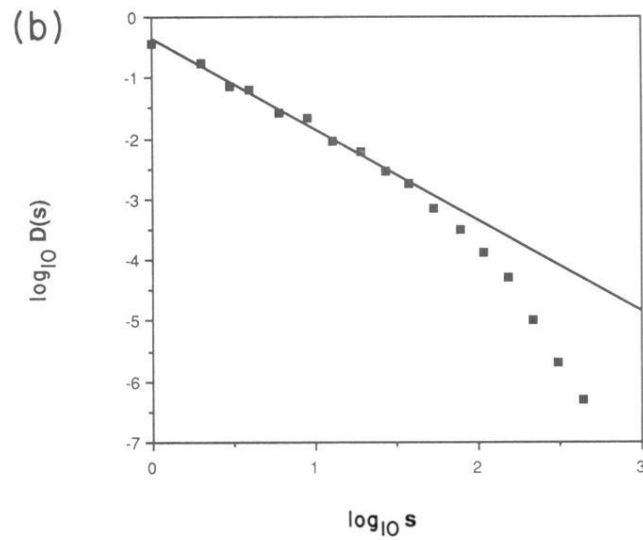


FIG. 2. (a) DLA cluster of mass $M=4000$. The surviving paths are shown as heavy lines. The rest of the cluster is a disconnected set of extinct branches. The surviving perimeter sites are shown in green, the surviving paths in blue, and the extinct branches in red. (b) The size distribution $D(s)$ of extinct branches, based on 40 clusters of mass $M=4000$ (totaling ~ 14000 extinct branches). The slope of the straight line is -1.50 and is fitted to the data for $3 \leq s \leq 45$. (c) $D(s)s^{1.5}$ vs sp^* based on 20 clusters of mass $M=400$ (\times), $M=1000$ (\square), and $M=4000$ (\blacktriangle). The data obtained for 40 clusters with mass $M=4000$ are also given (\triangle) for comparison. (d) $MD(t)$ vs t/M , symbols as in (c). The large scatter at small t is an effect of the broad distribution that only leaves a few number of lifetimes in the first "bins."