Power Law Scaling for a System of Interacting Units with Complex Internal Structure

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We study the dynamics of a system composed of interacting units each with a complex internal structure comprising many subunits and treat the case in which each subunit grows in a multiplicative manner. We propose a model for such systems in which the interaction among the units is treated in a mean field approximation and the interaction among subunits is nonlinear. We test the model and find agreement between our predictions and empirical results based on a large economics database spanning 20 years.   [S0031-9007(98)05355-1]

In the physical sciences, power law scaling is usually associated with critical behavior (thus requiring a particular set of parameter values), or with scale free growth processes [1]. For example, in the Ising model there is a particular value of the strength of the interaction between the units composing the system that generates correlations extending throughout the entire system and leads to power law distributions. In the social and biological sciences, there also appear examples of power law distributions (such as incomes [2], bird populations [3], and heart dynamics [4]). Although self-organized criticality has been the preferred explanation for these results, it is difficult to imagine that for all these diverse systems, the parameters controlling the dynamics spontaneously self-tune to their critical values.

In this Letter, we propose an alternative mechanism, in the spirit of scale free growth processes, that could explain how power law scaling in biological or social sciences can emerge even in the absence of critical dynamics. The guiding principles for our approach, to be justified below, are as follows: (i) The units composing the system have a complex evolving structure (e.g., the firms competing in an economy are composed of divisions, the cities in a country competing for the mobile population are composed of distinct neighborhoods, the population of some species living in a given ecosystem might be composed of groups living in different areas), and (ii) the size of the subunits composing each unit evolve according to a random multiplicative process.

Fortunately, for one of the examples listed above, there is a wealth of quantitative data, and here we focus on a large database giving the time evolution of the size of firms [5]. In an economy, the units composing the entire system are the competing firms. In general, these firms have a complex internal structure, with each firm composed of divisions (the subunits of each unit). It has been proposed that the evolution of a firm's size is described by a random multiplicative process with variance independent of the size, and that each firm can be viewed as a structureless unit [6]. However, later studies [7–10] reveal that the dynamics of real firms are not fully consistent with the simplified picture of Ref. [6].

We develop a model that dynamically builds a diversified, multidivisional structure, reproducing the fact that a typical firm passes through a series of changes in organization, growing from a single-product, single-plant firm, to a multidivisional, multiproduct firm [11]. The model reproduces a number of empirical observations for a wide range of values of parameters and provides a possible explanation for the robustness of the empirical results. Because of our encouraging results for the case of firm growth, our model may offer a generic approach to explain power law distributions in other complex systems.

The model, illustrated in Fig. 1, is defined as follows. A firm is created with a single division, which has a size $\xi_1(t = 0)$. The size of a firm $S \equiv \Sigma_i \xi_i(t)$ at time t is the sum of the sizes of the divisions $\xi_i(t)$ comprising the firm. We define a minimum size $S_{\min}$ below which a firm would not be economically viable, due to the competition between firms; $S_{\min}$ is a characteristic of the industry in which the firm operates. We assume that the size of each division i of the firm evolves according to a random multiplicative process [6]. We define

$$\Delta\xi_i(t) \equiv \xi_i(t)\eta_i(t), \tag{1}$$

where $\eta_i(t)$ is a Gaussian-distributed random variable with zero mean and standard deviation V independent of ξ_i. The divisions evolve as follows:

(i) If $\Delta\xi_i(t) < S_{\min}$, division i evolves by changing its size, and $\xi_i(t + 1) = \xi_i(t) + \Delta\xi_i(t)$. If its size becomes smaller than $S_{\min}$—i.e., if $\xi_i(t + 1) < S_{\min}$—then with probability p_a, division i is "absorbed" by division 1. Thus, the parameter p_a reflects the fact that when a division becomes very small it will no longer be viable due to the competition between firms.

(ii) If $\Delta\xi_i(t) > S_{\min}$, then with probability $(1 - p_f)$, we set $\xi_i(t + 1) = \xi_i(t) + \Delta\xi_i(t)$. With a probability p_f, division i does not change its size—so that
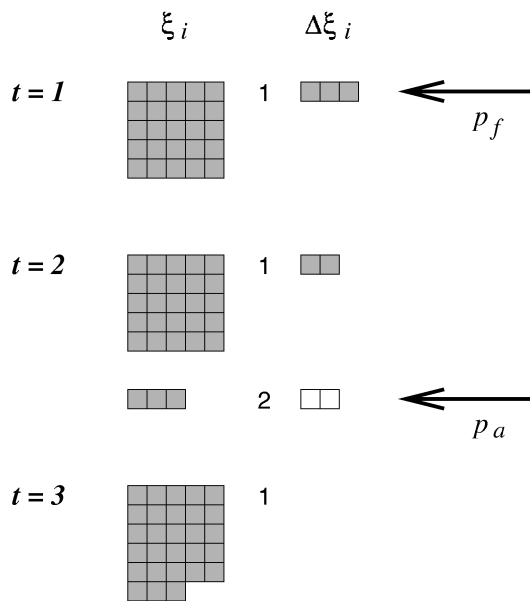
FIG. 1. Schematic representation of the time evolution of the size and structure of a firm. We choose $S_{min} = 2$, and $p_f = p_a = 1.0$. The first column of full squares represents the size ξ_i of each division, and the second column represents the corresponding change in size $\Delta\xi_i$. Empty squares represent negative growth and full squares positive growth. We assume, for this example, that the firm has initially one division of size $\xi_1 = 25$, represented by a 5×5 square. At $t = 1$, division 1 grows by $\Delta\xi_1 = 3$. A new division, numbered 2, is created because $\Delta\xi_1 > S_{min} = 2$, and the size of division 1 remains unchanged, so for $t = 2$, the firm has 2 divisions with sizes $\xi_1 = 25$ and $\xi_2 = 3$. Next, divisions ξ_1 and ξ_2 grow by 2 and -2, respectively. Division 2 is absorbed by division 1, since otherwise its size would become $\xi_2 = 3 - 2 = 1$ which is smaller than S_{min}. Thus, at time $t = 3$, the firm has only one division with size $\xi_1 = 25 + 2 + 1 = 28$. Note that if division 1 would be absorbed, then division 2 would absorb division 1 and would then be renumbered 1. If division 1 is absorbed and there are no more divisions left, the firm "dies."

$\xi_i(t + 1) = \xi_i(t)$—and an altogether new division j is created with size $\xi_j(t + 1) = \Delta\xi_i(t)$. Thus, the parameter p_f reflects the tendency to diversify: the larger is p_f, the more likely it is that new divisions are created.

The dynamics are thus controlled by three independent parameters: V, p_a, and p_f—S_{min} just sets the scale, so the results of the model do not depend on its value. We assume that there is a broad distribution of values of S_{min} in the system because firms in different activities will have different constraints.

In Fig. 2, we compare the predictions of the model for the distribution of firm sizes in the stationary state with the empirical data [10]. The stationary state is reached after approximately 10 "years," provided that new firms are created regularly. We define one "year" as ℓ iterations of our rules applied to each firm, and we find no significant dependence of the results on the value of ℓ for $\ell > 10$. We find similar results for a wide range of parameters: $V = 0.1$–0.2, $p_a = 0.01$–1, and $p_f = 0.1$–1.0.
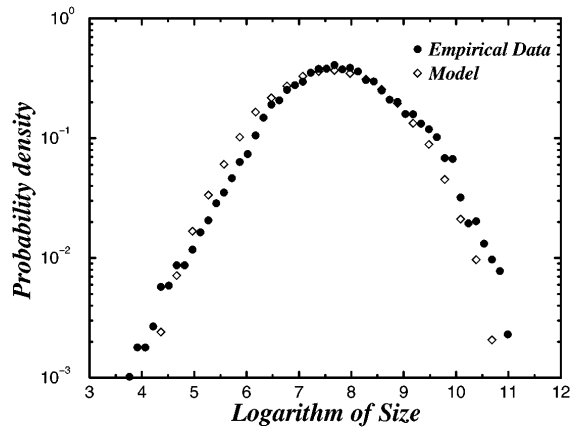
FIG. 2. Probability density of the logarithm of firm size for the model and for U.S. publicly traded manufacturing firms in the 1994 "COMPUSTAT" database. These results were obtained drawing $\log S_{min}$ from a Gaussian distribution with average value $\log(5 \times 10^5)$ and width $\mathcal{D} = 5$. Similar results would be obtained for other broad distributions of S_{min}. The numerical simulations were performed with parameters $V = 0.15$, $p_f = 0.8$, $p_a = 0.05$, and $\ell = 50$ (for these parameter values, the actual probability of a new division being created per division and per iteration is approximately 0.01).

It is common to study the logarithm of the one-year growth rate, $r_1 \equiv \ln R_1$, where $R_1 \equiv S(y + 1)/S(y)$, with $S(y)$ and $S(y + 1)$, are the sizes of the firm in the years y and $y + 1$. The empirical distribution of r_1 for firms with size S is, to first order approximation, consistent with an exponential form [10]

$$p(r_1|S) = \frac{1}{\sqrt{2}\,\sigma_1(S)}\,\exp\left(-\frac{\sqrt{2}\,|r_1 - \bar{r}_1|}{\sigma_1(S)}\right), \quad (2)$$

where $\bar{r}_1$ represents the average growth rate. Moreover, the standard deviation $\sigma_1(S)$ is consistent with a power law form

$$\sigma_1(S) \sim S^{-\beta}, \quad (3)$$

and for U.S. manufacturing firms, $\beta \approx 0.2$ [10]. We find that $p(r_1|S)$ is quite similar in form to the empirical results [10]. Figure 3(a) compares $\sigma_1(S)$ with the empirical data of Ref. [10]: for both, Eq. (3) holds with $\beta = 0.17 \pm 0.03$. Equations (2) and (3) allow us to scale the growth rate distributions for different firm sizes [Fig. 3(b)].

We next address the question of the structure of a given firm. To this end, we calculate the probability density $\rho_1(\xi_i|S)$ to find a division of size ξ_i in a firm of size S. For the model, we find that the distribution ρ_1 scales as a power law up to S^α and then it decays exponentially. Hence, we make the hypothesis that ρ_1 obeys the scaling relation

$$\rho_1(\xi_i|S) \sim S^{-\alpha} f_1(\xi_i/S^\alpha), \quad (4)$$

where $f_1(u) \sim u^\tau$ for $u \ll 1$ with $\tau \approx 2/3$. This hypothesis is confirmed by the scaling plot of Fig. 4(a). We find $\alpha = 0.66 \pm 0.05$ from plotting the average value of
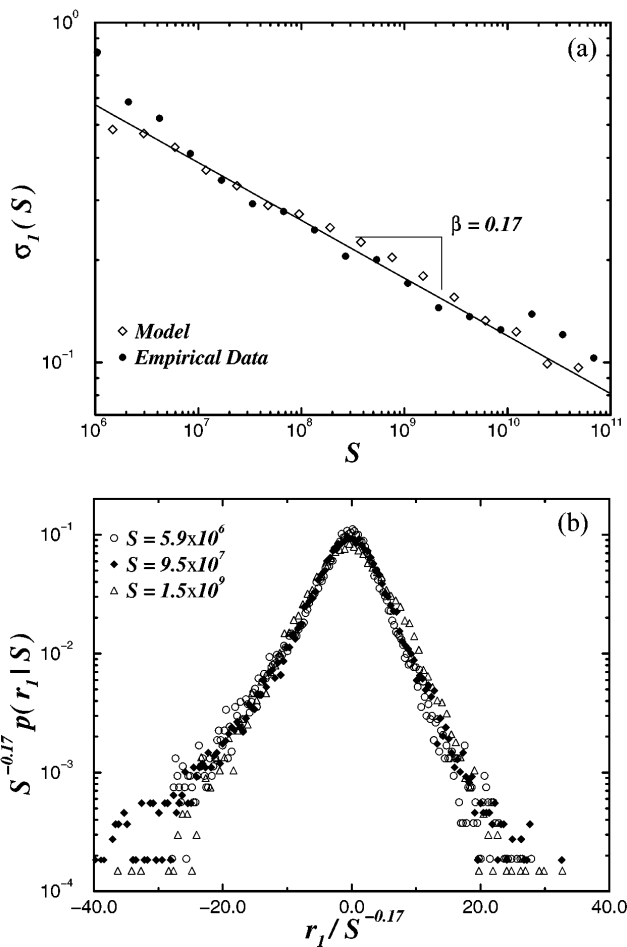
FIG. 3. (a) Dependence of the standard deviation of the growth rates on firm size. Shown are the predictions of the model and the empirical results. The values of the parameters are the same as in Fig. 2. The straight line with slope 0.17 is a least squares fit to the predictions of the model. (b) Probability density of one-year growth rates for different firm sizes plotted in scaled variables. The distributions are tent shaped, as for the empirical data [10], and consistent with an exponential distribution.

ξ_i against S. The same value of α leads to the best scaling plot.

Next, we make the hypothesis that the probability density $\rho_2(N|S)$ to find a firm with size S composed of N divisions obeys the scaling relation

$$\rho_2(N|S) \sim S^{-(1-\alpha)} f_2(N/S^{1-\alpha}). \qquad (5)$$

In writing (5), we use the fact that from (4) the characteristic size of a typical division scales as S^α, so that the typical number of divisions in a firm is $S/S^\alpha \sim S^{1-\alpha}$. Figure 4(b) shows that the results of the model are consistent with the scaling relation (5), with the same value of the scaling exponent α used in Fig. 4(a).

The results described by Eqs. (4) and (5) are in qualitative agreement with empirical studies [9] that show larger firms to be more diversified. Moreover, Eq. (5) states that the number of independent subunits in a firm of size S scales as $S^{1-\alpha}$. Since N does not change much during a
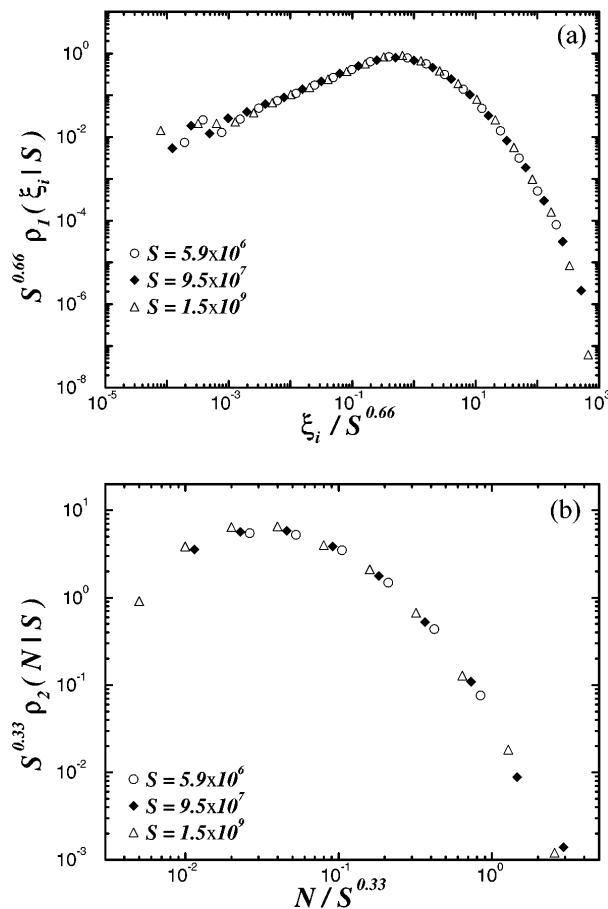

FIG. 4. (a) Data collapse of the conditional probability density ρ_1; the data fall onto a single curve corresponding to the scaling form (4). (b) Data collapse of the conditional probability density ρ_2; the data fall onto a single curve corresponding to the scaling form (5).

year and assuming that the subunits have similar sizes, we can apply the central limit theorem, from which it follows that $\sigma_1 \sim N^{-1/2}$, leading to the testable scaling law

$$\beta = (1 - \alpha)/2. \qquad (6)$$

For $\alpha = 0.66 \pm 0.05$, Eq. (6) predicts $\beta = 0.17 \pm 0.03$, in remarkable agreement with our independent calculation of β.

We find that the predictions of the model are only weakly sensitive to the parameter values, which perhaps is the reason why firms operating in quite different industries are described by very similar empirical laws. Accordingly, we conjecture that the scaling laws found for U.S. manufacturing firms [10] also hold for other countries, such as Japan, with $\beta \approx 0.2$; this conjecture is currently being tested with empirical data [12].

The present model rests on a small number of assumptions. The three key assumptions are as follows: (i) Firms tend to organize themselves into multiple divisions once they achieve a certain size. This assumption holds for many modern corporations [11]. (ii) There is a broad distribution of minimum scales in the economy. This assumption has also been verified empirically [8].

(iii) Growth rates of different divisions are independent of one another. For an economist, the latter is the stronger of these assumptions. However, we find that correlations in the growth rates of divisions within the same firm, even weak correlations, lead to $\beta \to 0$. Thus, we confirm that it is the assumption of independence among the growth rates that generates results similar to the empirical findings of Ref. [10].

There are two features of our results that are perhaps surprising. First, although firms in our model consist of independent divisions, we do not find $\beta = 1/2$. To understand why $\beta < 1/2$, suppose that the distribution of $s_m \equiv \ln S_{\min}$ is a Dirac-δ function. Although this assumption is unrealistic, it leads to an understanding of the underlying mechanisms in the model. In this case, it is a plausible assumption that the number of divisions will increase linearly with firm size, because the distribution of division sizes is narrow and confined between $S_{\min}$ and $S_{\min}/V$. This hypothesis is confirmed numerically, and we find (i) $\beta = 1/2$ and $\alpha = 0$, and (ii) that the distribution of the logarithm of firm sizes is still close to Gaussian, with a width $\mathcal{W}$ which is a function of the parameters of the model. Then, by integration of the distribution of the logarithm of firm sizes over s_m, we can estimate the value of β for the case of a broader distribution of s_m. Suppose that s_m follows some arbitrary distribution with width $\mathcal{D}$. Averaging $\sigma_1^2(S)$ over this distribution, we find $\beta = \mathcal{W}/2(\mathcal{D} + \mathcal{W})$. For a wide range of the values of the model's parameters, $\mathcal{D} > \mathcal{W}$, and we find that β is remarkably close to the empirical value $\beta \approx 0.2$.

Second, the distribution $p(r_1|S)$ is not Gaussian but "tent" shaped. We find this result arises from the integration of nearly Gaussian distributions of the growth rates over the distribution of $S_{\min}$. For large values of $|r_1|$, the saddle point approximation gives $p(r_1|S) \sim \exp(-\log^2 |r_1|)$, which decays slower than exponentially, in qualitative agreement with the model's predictions and with empirical observations. For $|r_1| \ll 1$, $p(r_1|S)$ is approximately Gaussian, while for intermediate values of $|r_1|$, the distribution decays exponentially. Our analytical predictions are in agreement with the model and with empirical results.

The model leads to a number of conclusions. First, it suggests the deviations in the empirical data from predictions of the random multiplicative process may be explained by (i) the diversification of firms, i.e., firms are made up of interacting subunits, and (ii) the fact that different industries have different underlying scales, i.e., there is a broad distribution of minimum scales for the survival of a unit (for example, a car manufacturer must be much larger than a software firm).

Second, the model suggests a possible explanation for the common occurrence of power law distributions in complex systems. Our results suggest that the empirically observed power law scaling does not require the system to be in a critical state, but rather can arise from an interplay between random multiplicative growth and the complex structure of the units composing the system. Here we addressed the case in which the interactions between the units can be treated in a "mean field" way through the imposition of a minimum size for the subunits. More general interactions may still lead to power law scaling, so our model may offer a framework for the study of complex systems.

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